GRADED CHEREDNIK ALGEBRA AND QUASI-INVARIANT DIFFERENTIAL FORMS

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Given a finite Coxeter group W acting on its finite dimensional reflection representation V, we consider the ring $\mathcal{D}(\Omega V) \rtimes W$ of equivariant differential operators on differential forms on V. After computing its Hochschild cohomology, we deduce the existence of a universal formal deformation of it, parametrized by the set of conjugation invariant functions on the set of reflections of W. We then introduce the graded Cherednik algebra, which represents an algebraic realization of that deformation. Using the deformed de Rham differential introduced by Dunkl and Opdam, we deform the standard Lie derivative operator and construct a Dunkl type embedding for the graded Cherednik algebra which extends the classical one. This embedding allows us to endow the graded Cherednik algebra with a natural differential. We study its cohomology and relate it to the singular polynomials of W. Finally, following the idea in the classical case, we investigate different notions of quasi-invariant differential forms and their relation to the spherical subalgebra of the graded Cherednik algebra.

BIOGRAPHICAL SKETCH

Youssef El Fassy Fihry was born in Rabat, Morocco on November 16, 1982, and moved to France in 2000 where he attended Université Pierre et Marie Curie and Ecole des Mines de Paris. He graduated from the latter in 2006 and moved to the United States where he was a graduate student in the Mathematics department at Cornell University. He worked under the supervision of Yuri Berest.

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CHAPTER 1

INTRODUCTION

1.1 Motivation

Over a field $\Bbbk$ of characteristic zero, there is a well developed algebraic theory of rings of differential operators on smooth varieties and their modules (see, e.g. [Bjö79]). If X is such a variety, the ring of differential operators on X, denoted by $\mathcal{D}(X)$, shares many properties with the Weyl algebra $A_n(\Bbbk) = \Bbbk[x_1, \ldots, x_n, \partial_i, \ldots, \partial_n]$, which is the ring of differential operators on the affine space $\Bbbk^n$. Just as is the case of the Weyl algebra, $\mathcal{D}(X)$ is generated by the coordinate ring $A = \mathcal{O}(X)$ of regular functions on X, and by the module $\operatorname{Der}_{\Bbbk} A$ of $\Bbbk$-linear derivations of A (see [MR01]). Furthermore, $\mathcal{D}(X)$ is a simple Noetherian domain, and a finitely generated algebra over $\Bbbk$ whose global homological dimension equals the dimension of X. Filtering $\mathcal{D}(X)$ by the order of differential operators and taking the associated graded rings, yields an isomorphism $\operatorname{gr} \mathcal{D}(X) \simeq \operatorname{Sym} \operatorname{Der} A$ which shows that $\mathcal{D}(X)$ can be viewed as a quantization of the ring of functions on the symplectic variety T^*X, the cotangent bundle of X. On the down side of this identification is that the deformation theory of these rings is trivial as they are often rigid (their Hochschild cohomology can be computed using the de Rham complex of X, see [WX98]).

In the situation where a finite group W acts on X, this action transfers to the ring $\mathcal{O}(X)$ of function on X and it is well known that the subring $\mathcal{O}(X)^W \subset \mathcal{O}(X)$ of functions on X which are invariant under the action of W is an affine algebra. Its geometric properties depend greatly on the nature of X and the action of W. If $\Bbbk = \mathbb{C}$ and X is a finite dimensional vector space V viewed as an affine space with ring of functions $\mathbb{C}[V] := \mathrm{Sym}(V^*)$, then the smoothness of $\mathbb{C}[V]^W$ is equivalent to W being generated by reflections ([Che55, ST54]), i.e. finite order elements which fix a hyperplane in V pointwise. In fact, in this case $\mathbb{C}[V]^W$ is a free polynomial algebra and hence geometrically rather uninteresting.

On the other hand, since W acts on $\mathbb{C}[V]$, we can extend this action to $\mathcal{D}(V) := \mathcal{D}(\mathbb{C}[V])$ and consider the rings $\mathcal{D}(V)^W$ and $\mathcal{D}(V) \rtimes W$ of invariant (resp. equivariant) differential operators on V. These algebras have a very non trivial deformation theory. Indeed, for W a complex reflection group acting on its finite dimensional reflection representation, the Hochschild cohomology of $\mathcal{D}(V) \rtimes W$ vanishes in degree 1 and 3, and $HH^2(\mathcal{D}(V) \rtimes W)$ is isomorphic to $\mathbb{C}[\Sigma]^W$, the space of functions on the set of reflections $\Sigma \subset W$, which are constant on the conjugacy classes (see [AFLS00]). This means that $\mathcal{D}(V) \rtimes W$ admits a universal family of formal deformations parametrized by $\mathbb{C}[\Sigma]^W$.

The question of whether one can realize universal deformations algebraically is a natural one, but often not a trivial one to answer. In this particular case, the rational Cherednik algebras $H_k = H_k(V, W)$, which can be viewed as degenerations of Cherednik's double affine Hecke algebras (see [Che05]), provide a realization of the universal deformations of $\mathcal{D}(V) \rtimes W$. If V_{reg} denotes the complement of the reflection hyperplanes in V, then H_k can be realized as the subalgebra of $\mathcal{D}(V_{reg}) \rtimes W$ generated by $\mathbb{C}[V]$, W, and certain differential-reflection operators introduced originally by Dunkl ([Dun89]).

These Dunkl operators commute and generate a subalgebra of H_k isomorphic to $\mathbb{C}[V^*]$. In fact, a PBW type theorem holds and $H_k \cong \mathbb{C}[V] \otimes \mathbb{C}W \otimes \mathbb{C}[V^*]$ as $\mathbb{C}[V]$-modules. In this decomposition each tensorand is actually a subalgebra of H_k and thus H_k is an algebra with triangular decomposition, similar to the triangular decomposition of the universal enveloping algebra of a simple complex Lie algebra $\mathfrak{g}$. It turns out that H_k has a very interesting representation theory which is in many ways similar to the classical represention theory of $\mathfrak{g}$. In fact, the representation theory of the rational Cherednik algebras has deep connections with, among other things, symplectic resolutions, algebraic combinatorics, completely integrable systems, generalized McKay correspondence, and the study of quasi-invariant polynomials and their rings of differential operators (see e.g. [BEG03, BC11, Eti07, Gor08]).

For finite Coxeter groups, the notion of a quasi-invariant polynomial was introduced by O. Chalykh and A. Veselov in [CV90]. Although quasi-invariants are a natural generalization of invariants, they first appeared in a slightly disguised form (as symbols of commuting differential operators). More recently, the algebras of quasi-invariants and associated varieties have been studied in [FV02, EG02a, BEG03] by means of representation theory and have found applications in other areas. For general complex reflection groups, the theory of quasi-invariants has been developed in [BC11].

Motivated by the fact that passing from the ring of function on V to the ring of differential forms on V amounts to passing from $\operatorname{Sym} V^*$ to $\underline{Sym}\, \mathbb{V}^*$, where $\mathbb{V}$ is the graded vector space $V \oplus V[1]$, we set to study what happens to the picture described above when one replaces the ring $\mathcal{D}(V)$ by the ring $\mathcal{D}(\Omega V)$ of differential operators on differential forms on V. Forming $\mathcal{D}(\Omega V) \rtimes W$, we study its structure and construct a universal deformation of it. Doing so, we obtain a graded modification of the rational Cherednik algebra which has a natural dg structure. We study its cohomology and relate it to the singular polynomials of W. We also investigate different notions of quasi-invariants for differential forms and their relationship to the graded Cherednik algebra. Finally, we point out that the differential graded Cherednik algebra can be viewed as an analog of the noncommutative Weil algebra in Lie theory (see [AM00]).

1.2 Statement of results

Given a finite reflection group W acting on its finite dimensional reflection representation V, one can make W act on $\mathbb{C}[V] = \operatorname{Sym} V^*$ and $\Omega V = \mathbb{C}[V] \otimes \bigwedge V^*$ naturally. This action transfers to the algebra $\mathcal{D}(\Omega V)$ of differential forms on ΩV. We form the algebra $\mathcal{D}(\Omega V) \rtimes W$ and study its deformations. In order to do so, we prove the following result.

Theorem. *Let W be a Coxeter group acting on its finite dimensional reflection representation V. The algebra $\mathcal{D}(\Omega V) \rtimes W$ is Morita equivalent to $\mathcal{D}(V) \rtimes W$ and thus, since Hochschild cohomology is invariant under Morita equivalence, $HH^\bullet(\mathcal{D}(\Omega V) \rtimes W) \cong HH^\bullet(\mathcal{D}(V) \rtimes W)$.*

In particular, we get the following corollary :

Corollary. $HH^2(\mathcal{D}(\Omega V) \rtimes W) = \mathbb{C}[\Sigma]^W$*, where $\Sigma \subset W$ is the subset of reflections, and* $HH^1(\mathcal{D}(\Omega V) \rtimes W) = HH^3(\mathcal{D}(\Omega V) \rtimes W) = 0$*. Hence, there exists a universal family of formal deformations of $\mathcal{D}(\Omega V) \rtimes W$ parametrized by $\mathbb{C}[\Sigma]^W$, the conjugation invariant functions on the set of reflections.*

We then focus on constructing the deformations of $\mathcal{D}(\Omega V) \rtimes W$ algebraically. Noticing that $\Omega = \underline{Sym}\,[\mathbb{V}^*]$, the graded symmetric algebra on the graded vector space $\mathbb{V} := V \oplus V[1]$, passing from $\mathcal{D}(V)$ to $\mathcal{D}(\Omega V)$ amounts to considering

$\mathcal{D}(\mathbb{V}) := \mathcal{D}\left(\underline{Sym}(\mathbb{V}^*)\right)$. We thus define the *graded Cherednik* algebra as a quotient of the graded tensor algebra

$$T(\mathbb{V} \oplus \mathbb{V}^*) \rtimes W$$

by reinterpreting the defining relations of the standard rational Cherednik algebra. Let us write $H_k = H_k(W)$ for the standard rational Cherednik algebra, and $Cl(V \oplus V^*)$ for the Clifford algebra on $V \oplus V^*$ with respect to the canonical symmetric pairing. We prove the following theorem:

Theorem. *The graded Cherednik algebra $\mathbb{H}_k$ is isomorphic as a graded algebra to*

$$H_k \otimes Cl(V \oplus V^*)$$

where H_k sits in degree 0, $V \subset Cl(V \oplus V^*)$ in degree -1 and $V^* \subset Cl(V \oplus V^*)$ in degree 1.

The graded equivalent of the standard theorems still holds and we have

Theorem. *The graded* Cherednik *algebra $\mathbb{H}_k$ has the following properties:*

(i) $\mathbb{H}_0 = \mathcal{D}(\Omega V) \rtimes W$.

(ii) PBW property: the linear map $\Omega V \otimes \mathbb{C}W \otimes \Omega V^* \to \mathbb{H}_k$ induced by multiplication in $\mathcal{H}_k$ is a ΩV-module isomorphism.

(iii) The family $\{\mathbb{H}_k\}$ is a universal deformation of $\mathbb{H}_0 = \mathcal{D}(\Omega V) \rtimes W$.

Similarly to the classical situation, the graded Cherednik algebra comes with a Dunkl-type embedding

$$\mathbb{H}_k \hookrightarrow \mathcal{D}(\Omega V_{reg}) \rtimes W$$

We construct this embedding and show that the localization isomorphism still holds:

Theorem. *There is an embedding* $\mathbb{H}_k \hookrightarrow \mathcal{D}(\Omega V_{reg}) \rtimes W$*. The set* $\{\delta^n, n \geq 0\}$ *is an Ore subset in* $\mathbb{H}_k$*, and the natural map* $\mathbb{H}_k[\delta^{-1}] \to \mathcal{D}(\Omega V_{reg}) \rtimes W$ *is an isomorphism of graded algebras.*

Most interestingly though, the Dunkl representation allows us to endow $\mathbb{H}_k$ with a natural structure of a differential graded algebra. More precisely, we show

Theorem. *The graded Cherednik algebra* $\mathbb{H}_k = H_k \otimes Cl(V \oplus V^*)$ *has a natural dg structure with differential given by*

$$\begin{aligned}
&d(s \otimes 1) = -s\alpha_s^\vee \otimes \alpha_s \\
&d(\xi \otimes 1) = 0, \quad d(x \otimes 1) = 1 \otimes x - \sum_{s \in \Sigma} k_s \langle x, \alpha_s^\vee \rangle (s \otimes \alpha_s) \\
&d(1 \otimes \xi) = \xi \otimes 1, \quad d(1 \otimes x) = 0
\end{aligned}$$

where $\{\alpha_s \in V^*, s \in \Sigma\}$ *and* $\{\alpha_s^\vee \in V, s \in \Sigma\}$ *define the reflection hyperplanes of* W*.*

We then study the cohomology of $\mathbb{H}_k$ and relate it to singular polynomials of W. Let $\mathrm{Irr}(W)$ be the set of isomorphism classes of irreducible representations of W and let τ be a $\mathbb{C}W$-module. We can make τ into a $\mathbb{C}[V^*] \rtimes W$-module by making V act trivially on τ. We can then form the following left H_k-module

$$M(\tau) := \mathrm{Ind}_{\mathbb{C}[V^*] \rtimes W}^{H_k} \tau = H_k \otimes_{\mathbb{C}[V^*] \rtimes W} \tau \tag{1.1}$$

When τ is irreducible, $M(\tau)$ is the *standard module of type* τ. Define now $\mathfrak{v}$ to be the commutative Lie algebra on the vector space V. Then $\mathfrak{v} = V \subset H_k$ acts on $M(\tau)$ and hence on $M(\tau) \otimes \tau^*$, and we can form $M(\tau)^{\mathfrak{v}} \otimes \tau^*$, which is the set of *singular polynomials* in representation τ, i.e elements in $M(\tau) \cong \mathbb{C}[V] \otimes \tau$ which are in the common kernel of *all* the Dunkl operators. We form the following conjecture.

Conjecture 1.2.1. The cohomology ring of $\mathbb{H}_k$ is

$$H^0(\mathbb{H}_k) \cong \bigoplus_{\tau \in \mathrm{Irr}(W)} M(\tau)^{\mathfrak{v}} \otimes \tau^*$$

$$H^i(\mathbb{H}_k) = 0, \; i \neq 0, \quad k \text{ integral}$$

In particular, if k is *regular* in all representations, i.e. if the common kernels of the Dunkl operators are trivial in all representations, then

$$H^0(\mathbb{H}_k) \simeq \mathbb{C}W$$

We prove that this conjecture holds in the rank 1 case, namely,

Theorem. *(Cohomology of $\mathbb{H}_k(\mathbb{Z}/2)$) Let $k \notin \mathbb{Z} + \frac{1}{2}$, then the following isomorphisms hold:*

$$H^{-1}\mathbb{H}_k = 0, \quad H^0\mathbb{H}_k \simeq \mathbb{C}[\mathbb{Z}/2], \quad H^1\mathbb{H}_k = 0$$

Finally, generalizing the ideas behind the classical case (see [BC11]), we conclude by introducing and studying the space of quasi-invariant differential forms. The standard definition suggests two different candidates for the notion of quasi-invariant differential forms. One approach leads to a boring ring with a Solomon type decomposition of the form $Q_k \otimes_{\mathbb{C}[V]^W} (\Omega V)^W \cong Q_k \otimes \bigwedge(df_1, \ldots, df_m)$, where $Q_k = Q_k(W)$ is the ring of standard quasi-invaraints, and $f_1, \ldots, f_m$ are fundamental invariants of W. The other is more interesting, yielding a ring $\mathcal{Q}_k$ which we we prove satisfies the following property, by passing to $\mathbb{C}W$-valued quasi-invariant differential forms.

Theorem 1.2.2. *Let $\boldsymbol{e} := \frac{1}{|W|}\sum_{w \in W} w$. Then $\mathcal{Q}_k$ is naturally a dg-module over the graded spherical sub-algebra $\boldsymbol{e}\mathbb{H}_k\boldsymbol{e}$, which acts on $\mathcal{Q}_k$ by invariant differential operators through the spherical Dunkl embedding* $\mathrm{Res} : \boldsymbol{e}\mathbb{H}_k\boldsymbol{e} \hookrightarrow \mathcal{D}(\Omega V_{reg})^W$.

CHAPTER 2

PRELIMINARIES

Throughout, $\Bbbk$ denotes a field with $\mathrm{char}(\Bbbk) \neq 2$. All algebras are assumed to be associative, with unit and defined over $\Bbbk$ (unless otherwise specified). By unadorned tensor products $\otimes$ we will always mean $\otimes_{\Bbbk}$, i.e. tensor products over $\Bbbk$.

2.1 Finite reflection groups and their invariants

2.1.1 Finite reflection groups

In this section we review the basic definition and properties of finite reflection groups. For a more complete introduction, see [Hum92] and [Ben93].

Let V be a finite dimensional $\Bbbk$-vector space. We define two distinguished classes of linear automorphisms of V.

Definition 2.1.1. An automorphism $s \in \mathrm{GL}_{\Bbbk}(V)$ is called

(i) a *pseudo- (or complex) reflection* if it has finite order $n_s > 1$ and there is a hyperplane $H_s \subset V$ that is pointwise fixed by s. Note that if s is diagonalizable, then this is equivalent to saying that all but one eigenvalues of s are equal to 1.

(ii) a *reflection* if it is a pseudo-reflection of order $n_s = 2$ which is in fact diagonalizable. In this case, the remaining eigenvalue is necessary equal to -1.

Example 2.1.2. Let $V = \mathbb{R}^n$ equipped with canonical Euclidean product $V \otimes V \mapsto \mathbb{R}$, $v \otimes w \mapsto v \cdot w$. For $\alpha \in V \setminus 0$, define $H_\alpha := (\alpha)^\perp$, where for $v \in V$, $v^\perp := \{w \in V : w \cdot v = 0\}$ is the orthogonal complement of v in V. Consider then the following linear operator on V:

$$s_\alpha(v) := v - \frac{(\alpha \cdot v)}{(\alpha \cdot \alpha)}\alpha, \quad v \in V$$

It is clear that by definition s_α leaves the hyperplane H_α pointwise fixed and sends α to $-\alpha$. Picking a basis of H_α and completing it with α, we see that s_α is diagonalizable, has order two and hence is a reflection. It is in fact the familiar orthogonal reflection with "mirror" hyperplane H_α.

Example 2.1.3. The use of the Euclidien inner product on $\mathbb{R}^n$ in the previous example is in fact unnecessary. Indeed, let V be any finite dimensional vector space over $\Bbbk = \mathbb{R}$ or $\mathbb{C}$ for example, and let V^* be its linear dual over K. We will write $\langle \cdot, \cdot \rangle$ for the canonical pairing $V^* \times V \to \Bbbk$. Then each $\alpha \in V^* \setminus \{0\}$ defines a reflection s_α on V as follows. Let $H_\alpha := \ker \alpha$, then H_α is a hyperplane in V. Choose $\alpha^\vee \in V$ such that $\langle \alpha, \alpha^\vee \rangle = 2$. Then it is easy to check that the linear operator s_α on V defined by

$$s_\alpha := \mathrm{Id}_V - \langle \alpha, \cdot \rangle \alpha^\vee$$

is a reflection which leaves H_α pointwise fixed and sends $\alpha^\vee$ to $-\alpha^\vee$. Note that, in a similar fashion, the operator $s_\alpha^* := \mathrm{Id}_{V^*} - \langle \cdot, \alpha^\vee \rangle \alpha$ is a reflection on V^*.

We will focus on subgroups of $\mathrm{GL}_\Bbbk(V)$ generated by (pseudo-) reflections.

Definition 2.1.4. A *finite reflection group* on V is a finite subgroup $W \subset \mathrm{GL}_\Bbbk(V)$ generated by pseudo-reflections, and a (finite) *Coxeter group* is a finite reflection group generated by actual reflections.

The previous definition is historically incorrect and not the right one for infinite Coxeter groups. In the case of finite groups though, it does coincide with a proper definition, as there is an equivalence between finite Coxeter groups and their faithful reflection representations.

For each (pseudo) reflection $s \in W$ with reflection hyperplane H_s, the pointwise stabilizer of H_s is a cyclic subgroup $W_{H_s} \subset W$ of order $n_s \geq 2$. Note that when $n_s = 2$, s is the only element in W_{H_s} of order 2.

Note that if W_1 is a finite reflection group on V_1 and W_2 is a finite reflection group on V_2, then $W = W_1 \times W_2$ is a finite reflection group on $V_1 \oplus V_2$. We say that W is *indecomposable* if it does not admit such a direct decomposition.

If $\Bbbk = \mathbb{R}$, then all pseudo-reflections are in fact reflections, and the classification of indecomposable finite groups generated by reflections was obtained in this case by H. S. M. Coxeter [Cox34]. There are 4 infinite families of indecomposable Coxeter groups (A_n, B_n, D_n and $I_2(m)$) and 6 exceptional groups ($E_6, E_7, E_8, F_4, H_3, H_4$).

Over the complex numbers $\Bbbk = \mathbb{C}$, the situation is much more complicated. A complete list of indecomposable complex reflection groups was given by G. C. Shephard and J. A. Todd in 1954: it includes 1 infinite family $G(m, p, n)$ depending on 3 positive integer parameters (with p dividing m), and 34 exceptional groups (see [ST54]). A. Clark and J. Ewing used the Shephard-Todd results to classify the groups generated by pseudo reflections over an arbitrary field of characteristic coprime to the group order (see [CE74]).

We provide a concrete realization of the main families of Coxeter groups.

Example 2.1.5 $(\mathbf{I_2(m), m \geq 3})$**.** Take $V = \mathbb{R}^2$ to be the Euclidean plane, and define D_m to be the dihedral group of order $2m$. Recall that D_m consists of the orthogonal transformations which preserve a regular m-sided polygon centered at the origin. D_m contains m rotations (through multiples of $\frac{2\pi}{m}$) and m reflections (about the "diagonals" of the polygon). Here "diagonal" means a line bisecting the polygon, joining two vertices or the midpoints of opposite sides if m is even, or joining a vertex to the midpoint of the opposite side if m is odd. Note that the rotations form a cyclic subgroup of index 2, generated by a rotation with angle $\frac{2\pi}{m}$. In fact, the group D_m is actually generated by reflections, because a rotation with angle $\frac{2\pi}{m}$ can be achieved as a product of two reflections relative to a pair of adjacent diagonals which meet at an angle of $\theta := \frac{\pi}{m}$. D_m is thus a reflection group. Note that the reflections form a single conjugacy class in D_m when m is odd, but form two classes when m is even.

Example 2.1.6 $(\mathbf{A_{n-1}, n \geq 2})$**.** Consider the symmetric group S_n. It can be thought of as a subgroup of the group $O(n, \mathbb{R})$ of $n \times n$ orthogonal matrices in the following way. Make a permutation act on $\mathbb{R}^n$ by permuting the standard basis vectors $e_1, \ldots, e_n$ (permute the subscripts). Observe that the transposition (ij) acts as a reflection, sending $e_i - e_j$ to its negative and fixing the orthogonal complement point wise, which consists of all vectors in $\mathbb{R}^n$ having their ith and jth components equal. Since S_n is generated by transpositions, it is a reflection group. Indeed, the transpositions $(i, i+1), 1 \leq i \leq n-1$ for example, are enough to generate S_n. Note that in this case, the transpositions are in fact the sole reflections belonging to S_n. When S_n acts on $\mathbb{R}^n$ in this way, it fixes all the points on the line spanned by $e_1 + \cdots + e_n$ (these are in fact clearly the only fixed points) and leaves stable the orthogonal complement, the hyperplane consisting of vectors whose coordinates add up to 0. Thus S_n also acts on an $(n-1)$-dimensional Euclidean space as a group generated by reflections, fixing no point except the origin. This accounts for the subscript $n-1$ in the label A_{n-1}.

Example 2.1.7 ($\mathbf{B_n, n \geq 2}$). Again let $V = \mathbb{R}^n$, so S_n acts on V as above. The additional reflections can be defined by sending a given basis element e_i to its negative and fixing all other elements $e_j, j \neq i$. These reflections, call them sign change reflections, generate a group of order 2^n isomorphic to $(\mathbb{Z}/2\mathbb{Z})^n$, which intersects S_n trivially and is normalized by S_n. Indeed, conjugating the reflection $e_i \mapsto -e_i$ by the action of a transposition in S_n yields another such sign change reflection. Thus the semidirect product of S_n and the group of sign change reflections yields a reflection group $(\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n$ of order $2^n n!$.

Example 2.1.8 ($\mathbf{D_n, n \geq 4}$). We can get yet another reflection group acting on $\mathbb{R}^n$ as a subgroup of index 2 in the group of type B_n just described above. S_n clearly normalizes the subgroup consisting of sign change reflections which involve an even number of signs, generated by the reflections $e_i + e_j \mapsto -(e_i + e_j),\ i \neq j$. Here again, the semidirect product is also a reflection group.

There are nice invariant-theoretic and geometric characterizations of finite reflection groups; we discuss them in the next section.

2.1.2 Polynomial invariants

Throughout this section, let V be a finite dimensional $\Bbbk$-vector space and $W \subset \mathrm{GL}_{\Bbbk}(V)$ is a finite group acting on V. We assume that the action of W on V is *faithful*, in other words that the kernel of the action is trivial.

Let V^* be the vector space dual to V and let $\Bbbk[V] := \mathrm{Sym}(V^*)$ denote the algebra of regular (i.e. polynomial) functions on V viewed as an affine space. This algebra carries a W-action coming from V. For $f \in \Bbbk[V]$, $g \in W$, and $v \in V$, we have

$$({}^g f)(v) = f(g^{-1}v).$$

Note that this is nothing but the natural action of W on V^*, extended diagonally to $\mathrm{Sym}(V^*)$.

Definition 2.1.9. The ring of (polynomial) *invariants* of W on V is defined by

$$\Bbbk[V]^W := \{f \in \Bbbk[V] : {}^g f = f \text{ for all } g \in W\}$$

This is clearly an algebra over $\Bbbk$ and in fact we have the following general fact:

Theorem 2.1.10. *Let W be a finite group acting on a finite dimensional $\Bbbk$-vector space V. Assume $\mathrm{char}(\Bbbk) = 0$ or $\mathrm{char}(\Bbbk)$ doesn't divide $|W|$, then $\Bbbk[V]^W$ is a finitely generated algebra, and $\mathrm{Spec}\,\Bbbk[V]^W$ is simply the orbit space V/W. In fact, it is an irreducible normal affine variety.*

Under the correspondence between commutative algebra and algebraic geometry, the inclusion $\Bbbk[V]^W \hookrightarrow \Bbbk[V]$ corresponds to the projection $V \to V/W$ which sends an element $v \in V$ to its orbit. The geometric properties of V/W – or equivalently, the algebraic properties of $\Bbbk[V]^W$ – depend deeply on the nature of W and its action on V. The following theorem shows that in the case of a finite reflection group and its reflection representation, the situation is unique and well understood.

Theorem 2.1.11 (Shephard-Todd-Chevalley-Serre, see [ST54], [Che55]). *Let V be a finite-dimensional faithful representation of a finite group W over a field $\Bbbk$. Assume that either* $\mathrm{char}(\Bbbk) = 0$ *or* $\mathrm{char}(\Bbbk)$ *is coprime to $|W|$. Then the following properties are equivalent:*

(i) W is generated by pseudo reflections.

(ii) $\Bbbk[V]$ is a free module over $\Bbbk[V]^W$.

(iii) $\Bbbk[V]^W$ is a free polynomial algebra of the form $\Bbbk[p_1, \ldots, p_n]$, where the p_i's are homogeneous polynomials of degree d_i.

(iv) The orbit space V/W is smooth.

(v) The projection map $V \to V/W$ is flat.

Over the complex numbers, this theorem was originally proved by Shephard and Todd [ST54] using their classification of pseudo-reflection groups. Later Chevalley [Che55] gave a proof in the real case without using the classification, and Serre extended Chevalley's argument to the complex case.

Note that while the generators $p_1, \ldots p_n$ in (iii) are not uniquely determined by W, their degrees d_i are and are thus called the *fundamental degrees* of W.

We conclude this section with another interesting result.

Theorem 2.1.12 ([Wal93], Theorem 2.2). *Let W be an indecomposable finite Coxeter group with reflection representation V. Assume that W is not of type E6, E7 or E8. Then the algebra $\mathbb{C}[V \oplus V^*]^W$ is generated as a Poisson algebra by $\mathbb{C}[V]^W$ and $\mathbb{C}[V^*]^W$.*

While one might hope that the previous theorem holds for *any* Coxeter group in general, but it does not (see [Wal93] for a counter-example). It can be replaced by the following weaker result which holds in full generality for any finite complex reflection group.

Theorem 2.1.13 ([BEG03], Lemma 4.6)**.** *Let W a finite reflection group with reflection representation the finite dimensional vector space V, and let $V_{reg} := V \setminus \bigcup_H H$, where H runs over the reflection hyperplanes in V.*

Then the algebra $\mathbb{C}[V_{reg} \times V^*]^W$ of regular functions on the symplectic manifold $(V_{reg} \times V^*)/W$ is generated by $\mathbb{C}[V_{reg}]^W$ and $\mathbb{C}[V^*]^W$ as a Poisson algebra.

2.1.3 Quasi-invariants

Closely related to the notion of invariants of a reflection group is that of quasi-invariants. The idea is to relax to requirement that ${}^g f = f$ and only ask for the equality modulo certain ideals in $\Bbbk[V]$.

For (finite) Coxeter groups, the notion of a quasi-invariant polynomial was introduced by O. Chalykh and A. Veselov in [CV90]. Although quasi-invariants are a natural generalization of invariants, they first appeared in a slightly disguised form (as symbols of commuting differential operators). More recently, the algebras of quasi-invariants and associated varieties have been studied in [FV02, EG02a, BEG03] by means of representation theory and have found applications in other areas. We recall here the definition of quasi-invariants for an arbitrary complex reflection group as introduced in [BC11].

The quasi-invariants form family of submodules of $\Bbbk[V]$ depending on some parameters that interpolate between $\Bbbk[V]^W$ and $\Bbbk[V]$. These submodules are defined for integral values of those parameters and can be interpreted as torsion-free coherent sheaves on certain (singular) algebraic varieties. The ring of *invariant* differential operators on such a variety turns out to be isomorphic to a spherical subalgebra U_k, and the modules of quasi-invariants become (via this isomorphism) objects of (the spherical analogue of) category $\mathcal{O}_k$. For a very well written summary of the subject and examples, [ES02].

We start by defining quasi-invariants for Coxeter groups, as the definition for complex reflection groups is more technical and less intuitive at first. From now on, we assume that $\Bbbk = \mathbb{C}$.

Let W be a finite reflection group acting on its finite dimensional reflection representation V and recall the definition of the algebras of regular functions on V and V^*, which are $\mathbb{C}[V] = \mathrm{Sym}(V^*)$ and $\mathbb{C}[V^*] = \mathrm{Sym}(V)$. Define $\mathcal{A}$ to be the set of all reflection hyperplanes, fixed by the generating reflections of W. For each reflection hyperplane $H \in \mathcal{H}$, let $\alpha_H \in V^*$ be such that $H = \ker \alpha_H$. Finally, for a multiplicity function $k : \mathcal{A} \to \mathbb{N}$ constant on the conjugaty classes of W, we make the following definition in the case where W is a Coxeter group:

Definition 2.1.14 ([CV90])**.** The set of *k-quasi-invariants* of W on V is

$$Q_k(W) := \left\{ q \in \mathbb{C}[V] : {}^{s_H}q \equiv q \mod \langle \alpha_H \rangle^{2k_H+1} \text{ for all } H \in \mathcal{A} \right\}$$

where $\langle \alpha_H \rangle$ is the ideal generated by α_H inside of $\mathbb{C}[V]$, and s_H is the unique element in W_H of order 2 and determinant -1.

For a complex reflection group, the previous definition is generalized as follows. First, notice that for a given reflection s, the condition ${}^{s}q \equiv q \mod \langle \alpha_s \rangle^{2k_s+1}$ is equivalent to $\frac{1}{2}(1-s)q \equiv 0 \mod \langle \alpha_s \rangle^{2k_s+1}$. Following this observation, we form the following definition:

Definition 2.1.15. For W a complex reflection group and $s \in W$ a reflection of order n_s with reflection hyperplane H_s we define the idempotents

$$e_{H_s,i} := \frac{1}{n_s} \sum_{w \in W_{H_s}} \det{}_V(w)^{-i} w \in \mathbb{C}W_{H_s} \subset \mathbb{C}W, \ i = 0, 1, \ldots, n_s - 1$$

Example 2.1.16. If $s \in W$ is a real reflection, then $n_s = 2$ and we get the two primitive idempotents

$$e_{H_s,0} = \frac{1}{2}(1+s), \quad e_{H_s,1} = \frac{1}{2}(1-s)$$

Let now $k := \{k_{H,i}\}_{H \in \mathcal{A}, i=0\ldots n_H-1}$, where k is chosen to be constant on the conjugacy classes of reflections. We also assume that $k_{H,0} = 0$ for all $H \in \mathcal{A}$. We are now ready for the following definition:

Definition 2.1.17 ([BC11])**.** For an arbitrary complex reflection group W, the set of quasi-invariants $Q_k \subset \mathbb{C}[V]$ is defined by

$$Q_k(W) := \left\{ q \in \mathbb{C}[V] : e_{H,-i}(q) \equiv 0 \mod \langle \alpha_s \rangle^{n_H k_{H,i}}, \ \forall 0 \leq i \leq n_H - 1, H \in \mathcal{A} \right\}$$

Note that when W is a Coxeter group, this definition agrees with the previous one because $k_{H,0} = 0$.

We now list some elementary properties of $Q_k(W)$:

Theorem 2.1.18. *Let W be a finite reflection group with reflection representation the finite dimensional vector space V, and let $Q_k(W)$ be it set of quasi-invariants. The following properties hold:*

(i) $\mathbb{C}[V]^W \subset Q_k \subset \mathbb{C}[V]$, $Q_0 = \mathbb{C}[V]$ *and* $Q_k \subset Q_{k'}$ *when* $k \geq k'$, *and* $\cap_k Q_k = \mathbb{C}[V]^W$.

(ii) If W *is a Coxeter group, then* Q_k *is a subring of* $\mathbb{C}[V]$, *and the fraction field of* Q_k *is equal to* $\mathbb{C}(V)$.

(iii) Q_k *is a finite* $\mathbb{C}[V]^W$*-module and a finitely generated algebra.*

(iv) If W *is a Coxeter group,* $\mathbb{C}[V]$ *is a finite* Q_k*-module.*

With the notation $Q_\infty := \mathbb{C}[V]^W$, motivated by the previous proposition, the family of rings of quasi-invariants Q_k now appear as an interpolation between $Q_0 = \mathbb{C}[V]$ and $Q_\infty = \mathbb{C}[V]^W$.

Example 2.1.19. Let $\mathbb{C}[V] = \mathbb{C}[x]$ and $W = \mathbb{Z}/2$ acting by the sign representation. Then an easy computation shows that $Q_k = \mathbb{C}[x^2] \oplus x^{2k+1}\mathbb{C}[x^2]$, and that it satisfies all the properties listed above. Note that Q_k is isomorphic to the ring of regular functions of the rational cuspidal curve $y^2 = x^{2k+1}$ in $\mathbb{C}^2$.

Furthermore, recall that for a finite reflection group W, the ring of invariants $\mathbb{C}[V]^W$ satisfies the properties that $\mathbb{C}[V]$ is a free $\mathbb{C}[V]^W$-module of rank $|W|$. Rephrasing this in term of quasi-invariants says that Q_0 is a free Q_∞-module. Obviously, Q_∞ is also a free module over itself. It light of this, it is then natural to ask whether this freeness property holds for Q_k in general. The answer to that comes in the following theorem:

Theorem 2.1.20. *The space of quasi-invariants* Q_k *is a free* $Q_\infty = \mathbb{C}[V]^W$*-module of rank* $|W|$, *for all* k.

In full generality, for an arbitrary complex reflection group, this theorem was proven in [BC11]. In the case of Coxeter groups, it was conjectured in [FV02] and proved in [EG02a] and [BEG03]. The original proof in [EG02a] is shorter, but the one in [BEG03] is more conceptual, and relies on the fact that Q_k is a module over eH_ke, the spherical sub-algebra of the rational Cherednik algebra of W. In fact, one can show that Q_k is a module in the category $\mathcal{O}(eH_ke)$ of finitely generated eH_ke-module on which $\mathbb{C}[V^*]$ acts locally nilpotently. The result then follows from studying the objects of that category.

2.2 Differential operators

The goal of this section is to recall some basic facts about differential operators on commutative algebras. A more complete introduction can be found in [MR01].

Roughly speaking, a differential operator is a mapping, typically understood to be linear, that transforms a function into another function by means of partial derivatives and multiplication by other functions. In the familiar context of multivariable calculus, a differential operator is commonly understood to be a linear transformation P of the space $\mathcal{C}^\infty(\mathbb{R}^n)$ of smooth functions of n variables, having the form:

$$P = \sum_{i_1,\dots,i_n} f_{i_1\dots i_n}(x_1,\dots,x_n)\frac{\partial^{i_1+\dots+i_n}}{\partial x_1^{i_i}\cdots\partial x_n^{i_n}} \tag{2.1}$$

where $(i_1,\dots,i_n)$ runs over all multi-indices in $\mathbb{N}^n$, the $f_{i_1},\dots f_{i_n}$ are smooth functions on $\mathbb{R}^n$, and the $\frac{\partial}{\partial x_i}$'s stand for the standard partial derivatives with respect to x_i. The action of such an operator on $g \in \mathcal{C}^\infty(\mathbb{R}^n)$ yields the function:

$$P[g] = \sum_{i_1,\dots,i_n} f_{i_1\dots i_n}(x_1,\dots,x_n)\frac{\partial^{i_1+\dots+i_n}g}{\partial x_1^{i_i}\cdots\partial x_n^{i_n}}$$

where we write $P[g]$ for the action of the operator P on g.

The set of all such operators is usually denoted by $\mathcal{D}(\mathbb{R}^n)$ and is clearly an algebra over $\mathbb{R}$, where multiplication is given by composition. Amongst these differential operators, there are included multiplication operator by smooth functions. Note that $\mathcal{D}(\mathbb{R}^n)$ is not a commutative algebra as for example

$$\left[\frac{\partial}{\partial x_1}, x_1\right][g] := \frac{\partial}{\partial x_1}(x_1 g) - x_1\frac{\partial g}{\partial x_1} = g + x_1\frac{\partial g}{\partial x_1} - x_1\frac{\partial g}{\partial x_1} = g \neq 0$$

Another thing worth mentioning is that certain operators like $\frac{\partial}{\partial x_1}$ or even $x_2\frac{\partial}{\partial x_1}$ satisfy the *Leibniz rule*, also known as the product rule, while others like $\frac{\partial^2}{\partial x_1^2}$ don't. More precisely, $P \in \mathcal{D}(\mathbb{R}^n)$ satisfies the Leibniz rule if for $f, g \in \mathcal{C}^\infty(\mathbb{R}^n)$, we have:

$$P[fg] = P[f]g + fP[g]$$

It is easy to see that $\mathcal{D}(\mathbb{R}^n)$ is generated as an algebra over $\mathbb{R}$ by multiplication operators and partial derivatives. In other words, every differential operator in $\mathcal{D}(\mathbb{R}^n)$ can be written using combinations of multiplication operators and operators that satisfy the Leibniz rule, also called *derivations*.

On the other hand, for $P \in \mathcal{D}(\mathbb{R}^n)$ as in (2.1), let the degree of P be the integer $\deg P := \max\{i_1 + \cdots + i_n : f_{i_1\cdots i_n} \neq 0\}$. Then given $P \in \mathcal{D}(\mathbb{R}^n)$ and $g \in \mathcal{C}^\infty(\mathbb{R}^n)$, the operator $[g, P] := gP - Pg$ is also a differential operator, but an easy computation shows that $\deg[g, P] \leq \deg P - 1$.

These two fundamental observations lead to two potentially different ways to generalize the notion of differential operators to an arbitrary commutative algebra A. We first introduce the notion of derivation ring. While natural, this approach has some limitations as we will see. Next we introduce the spaces of differential operators following the Grothendieck definition and show that in the case where the algebra A is regular, the two approaches agree.

2.2.1 Derivation rings

Let A be a commutative $\Bbbk$-algebra, and M and N two A-modules. The A-module structure on M and N allow us to equip the vector space $\mathrm{Hom}_{\Bbbk}(M, N)$ of $\Bbbk$-linear morphisms from M to N with both a left and right natural A-module structure. More precisely, for $\phi \in \mathrm{Hom}_{\Bbbk}(M, N)$, $m \in M$ and $a \in A$:

$$(a\phi)(m) := a\phi(m), \qquad (\phi a)(m) := \phi(am)$$

These two actions are compatible and $\mathrm{Hom}_{\Bbbk}(M, N)$ becomes a A-bimodule. A being commutative, this is equivalent to being an $A \otimes_{\Bbbk} A$-module as follows: for $\phi \in \mathrm{Hom}_{\Bbbk}(M, N)$ and $a, b \in A$, $(a \otimes b)\phi = a\phi b$. With this identification $[a, \phi] := a\phi - \phi a = (a \otimes 1 - 1 \otimes a)\phi$.

Definition 2.2.1. Let M be an A-module and $\theta \in \mathrm{Hom}_{\Bbbk}(A, M)$. We call θ a *derivation* of A with values in M if it satisfies the Leibniz rule. More precisely θ is a derivation of A if

$$\theta(ab) = \theta(a)b + \theta(b)a \quad a, b \in A$$

Write $\mathrm{Der}_{\Bbbk}(A, M)$ for the set of such derivations. This is a module over A, using the left multiplication coming from $\mathrm{Hom}_{\Bbbk}(A, M)$.

There is a *universal* derivation from A to a module and we describe it in the following. We start with a definition.

Definition 2.2.2. Let A be a commutative $\Bbbk$-algebra and let F be the free A-module generated by the elements da for $a \in A$.

The module of (Kähler) differentials $\Omega^1_{\Bbbk} A = \Omega^1 A$ of A is the quotient of F by the submodule generated by the following relations:

$$d\lambda = 0, \quad d(a+b) = da + db, \quad d(ab) = d(a)b + d(b)a \quad a, b \in A,\ \lambda \in \Bbbk$$

There is a natural map $d : A \to \Omega^1 A$ that sends $a \in A$ to the element da. This is a universal derivation in the following sense:

Proposition 2.2.3. *Let A, $\Omega^1 A$ be as above and let M be an A-module. Then*

(i) Given any derivation $\theta \in \mathrm{Der}(A, M)$, there exists a unique $\phi \in \mathrm{Hom}_A(\Omega^1 A, M)$ such that $\theta = \phi d$. In fact the map $\mathrm{Hom}_A(\Omega^1 A, M) \to \mathrm{Der}(A, M)$ given by $\phi \mapsto \phi d$ is an isomorphism of A-modules.

(ii) As A-modules, we have an isomorphism $\mathrm{Der}(A) \simeq \mathrm{Hom}_A(\Omega^1 A, A) =: (\Omega^1 A)^*$.

We now pose the following definition:

Definition 2.2.4 (Derivation ring)**.** Let A be a commutative algebra. The derivation ring of $\Delta(A)$ of A is the subalgebra of $\mathrm{End}_{\Bbbk}(A)$ generated by left multiplication by elements in A, and derivations $\theta \in \mathrm{Der}(A)$.

Example 2.2.5. Let $A = \Bbbk[x_1, \ldots, x_n]$, then $\mathrm{Der}(A)$ is the free A-module generated by the derivations ∂_i defined on generators by $\partial_i(x_j) = \delta_{ij}$. One then easily checks that $[\partial_i, x_j] = \delta_{ij}$, while $[\partial_i, \partial_j] = 0$ for $i \neq j$. In other words, $\Delta(A) = A_n(\mathbb{C})$ the Weyl algebra in n variables.

Example 2.2.6 ([MR01] 15.1.17)**.** In general, if $A = \mathbb{C}[x_1, \ldots, x_n]/\langle f_1, \ldots, f_m \rangle$, then

- $\Omega^1 A$ is generated by $dx_1, \ldots, dx_n$, with relations $\sum_j \frac{\partial f_i}{\partial x_j} dx_j = 0$ for all i.
- $\mathrm{Der}(A)$ is finitely generated, and there is a surjection $\{\delta \in \mathrm{Der}(\mathbb{C}[x_1, \ldots, x_n]) : \delta(I) \subset I\} \to \mathrm{Der}(A)$, where $I = \langle f_1, \ldots, f_n \rangle$.

There is a natural filtration on $\Delta(A)$ coming from giving elements in $\mathrm{Der}(A)$ degree 1. We have then the following result:

Proposition 2.2.7 ([MR01] 15.1.19)**.** $\mathrm{gr}\,\Delta(A)$ *is a commutative algebra. In fact, there is a surjection* $\mathrm{Sym}(\Omega^1 A)^* \to \mathrm{gr}\,\Delta(A)$

Recall that a commutative Noetherian integral domain A with maximal ideal $\mathfrak{m}$ is said to be *regular* if $\dim_{\Bbbk} \mathfrak{m}/\mathfrak{m}^2$ is equal to the Krull dimension of A. A connected affine variety $\mathrm{Spec}(A)$ is said to be *regular* (or smooth), if at each point $\mathfrak{m} \in \mathrm{Specm}(A)$, the local ring $A_{\mathfrak{m}}$ is regular.

Theorem 2.2.8 ([MR01] 15.3.8)**.** *The following conditions are equivalent:*

(i) A *is regular.*

(ii) $\Delta(A)$ *is simple.*

(iii) A *is simple as a* $\Delta(A)$*-module*

(iv) $\Omega^1 A$ *is a projective* A*-module.*

2.2.2 Rings of differential operators

Let $\mu : A \otimes_{\Bbbk} A \to A$ be the multiplication map, and $J \subset A \otimes_{\Bbbk} A$ its kernel. We are now ready for the following definition:

Definition 2.2.9. Let A be a commutative $\Bbbk$-algebra and M, N two A-modules. The space of $\Bbbk$-linear differential operators from M to N of order at most n is defined by

$$\mathcal{D}_A^n(M,N) = \left\{\theta \in \operatorname{Hom}_{\Bbbk}(M,N) : J^{n+1}\theta = 0\right\}$$

where $J^0 := A \otimes_{\Bbbk} A$.

Set $\mathcal{D}_A(M,N) := \bigcup_{n=0}^{\infty} \mathcal{D}_A^n(M,N)$, the filtered space of $\Bbbk$-linear differential operators from M to N and write $\mathcal{D}_A(M) := \mathcal{D}_A(M,M)$. We will drop the subscript A whenever there can be no confusion.

Before we move on to listing more properties of $\mathcal{D}(M,N)$, notice that $J = \ker\mu$ is generated by the elements $\Delta(a) := a \otimes 1 - 1 \otimes a$. The following proposition gives equivalent definitions for $D(M,N)$ which are more tractable and look more familiar. Set $\mathcal{D}^{-1}(M,N) := 0$, then

Proposition 2.2.10. *The following are equivalent for $\theta \in \operatorname{Hom}_{\Bbbk}(M,N)$:*

(i) $\theta \in \mathcal{D}^n(M,N)$*, for some* $n \geq 0$.

(ii) $\forall a \in A,\ [a,\theta] \in \mathcal{D}^{n-1}(M,N)$

(iii) $\forall a_0,\dots,a_n \in A,\ [a_0,[a_1,\dots[a_n,\theta]]] = 0$

Proof. This is clear once one realizes that $J\theta = 0$ is equivalent to $[a,\theta] = 0$ for all $a \in A$, and that J^{n+1} is generated by products of $n+1$ elements in J. $\square$

Given three modules M, N and L, the composition of morphisms gives rise to maps

$$\mathcal{D}^n(M,N) \otimes_{\Bbbk} \mathcal{D}^n(N,L) \to \mathcal{D}^n(M,L)$$

In the special case when $M = N = L$, we see that $\mathcal{D}(M)$ is a $\Bbbk$-subalgebra of $\operatorname{End}_{\Bbbk}(M)$. In general, composition of morphisms makes $D(M,N)$ a left $\mathcal{D}(N)$-module and a right $\mathcal{D}(M)$-module, and we have a bimodule map $\mathcal{D}(M,N) \otimes_{\mathcal{D}(M)} \mathcal{D}(L,M) \to \mathcal{D}(L,N)$.

We focus now on the algebra of differential operators $\mathcal{D}(A)$. There is an obvious relationship between $\mathcal{D}(A)$ and the derivation ring $\Delta(A)$ as defined in the previous section. Indeed, just as in the multivariable calculus situation, we have the following immediate result:

Proposition 2.2.11. *Let A be a commutative $\Bbbk$-algebra, then the following properties hold:*

(i) $\mathcal{D}^1(A) = A \oplus \operatorname{Der}(A)$

(ii) $\mathcal{D}(A)$ *is a filtered ring which contains* $\Delta(A)$ *as a filtered subring.*

Finally, we conclude this section with the following theorem, whose proof requires quite a bit of work.

Theorem 2.2.12 ([MR01] 15.5.6)**.** *Let A be regular $\Bbbk$-algebra, then $\mathcal{D}(A) = \Delta(A)$. In particular*

(i) $\mathcal{D}(A)$ *is a simple Noetherian ring.*

(ii) $\operatorname{gr} \mathcal{D}(A) \simeq \operatorname{Sym} \operatorname{Der}(A)$

(ii) $\mathcal{D}(A)$ *is generated as an algebra by A in filtered degree 0 and* $\operatorname{Der}(A)$ *in filtered degree 1.*

2.3 Deformation theory

2.3.1 Formal deformations of associative algebras

Let $\Bbbk$ be a field and $K = \Bbbk[\![\epsilon_1, \ldots, \epsilon_n]\!]$ the algebra of formal power series in n variables. Let $\mathfrak{m} = (\epsilon_1, \ldots, \epsilon_n)$ the maximal ideal in K.

Definition 2.3.1. Let A be an algebra over $\Bbbk$. A (flat) *formal n-parameter deformation* of A is an algebra A_ϵ over K which satisfies the following properties:

(i) A_ϵ is isomorphic as a K-module to $A[\![\epsilon_1, \ldots, \epsilon_n]\!]$.

(ii) There is an algebra isomorphism $\eta_0 : A_\epsilon/\mathfrak{m}A_\epsilon \xrightarrow{\sim} A$

We will restrict our attention in what follows to one-parameter deformations, in which case the previous definition unravels to the following one:

Definition 2.3.2. A *formal deformation* of A is a $\Bbbk[\![\epsilon]\!]$-bilinear multiplication law $m_\epsilon : A[\![\epsilon]\!] \otimes_{\Bbbk[\![\epsilon]\!]} A[\![\epsilon]\!] \to A[\![\epsilon]\!]$ on the space $A_\epsilon := A[\![\epsilon]\!]$ of formal power series in the variable ϵ with coefficients in A, satisfying the following properties:

(i) $m_\epsilon(a, b) = a \cdot b + m_1(a, b)\epsilon + m_2(a, b)\epsilon^2 + \cdots$ for $a, b \in A$, where $a \cdot b$ is the original multiplication in A.

(ii) m_ϵ is associative, in other words: $m_\epsilon(m_\epsilon(a, b), c) = m_\epsilon(a, m_\epsilon(b, c))$ for $a, b, c \in A$

If m_ϵ is only associative modulo $\epsilon^2 A[\![\epsilon]\!]$, then we say that A_ϵ is an *infinitesimal* or first order deformation of A. Similarly, one can define second, third, . . . order deformations.

It is clear that an order n deformation of A is equivalent to the data of a product

$$\star : A[\epsilon]/\epsilon^n \otimes_{\Bbbk[\epsilon]/\epsilon^n} A[\epsilon]/\epsilon^n \to A[\epsilon]/\epsilon^n$$

such that $a \star b = ab \mod \epsilon$ for $a, b \in A$.

Sometimes, an algebra A is endowed with extra structure and one is interested in deforming A in a way that is compatible with that structure. We are here mainly interested in deforming Poisson algebras, and we now recall their definition:

Definition 2.3.3. A Poisson algebra is a commutative algebra A equipped with a Lie bracket $\{-,-\} : A \times A \to A$ satisfying the Leibniz identity:

$$\{ab, c\} = a\{b, c\} + b\{a, c\},\ \ a, b, c \in A$$

We then form the definition:

Definition 2.3.4 (Deformation quantization)**.** A *deformation quantization* of a (commutative) Poisson algebra A is a one-parameter formal deformation $(A[\![\epsilon]\!], m_\epsilon)$ which is compatible with the Poisson structure:

$$m_\epsilon(a, b) \equiv ab \mod \epsilon, \quad \frac{m_\epsilon(a, b) - m_\epsilon(b, a)}{\epsilon} \equiv \{a, b\} \mod \epsilon, \quad \forall a, b \in A \quad (2.2)$$

2.3.2 Universal deformations and Hochschild cohomology

We now briefly explain how Hochschild cohomology relates to infinitesimal deformations (c.f. [Wei95]). This deformation-theoretical interpretation of Hochschild cohomology is due to M. Gerstenhaber [Ger64].

Let A be an associative algebra and M an A-bimodule. The associated Hochschild complex $C^\bullet(A,M)$ is defined as follows: $C^n(A,M)$ is the space of $\Bbbk$-linear maps $A^{\otimes n} \to M$ and the differential d is defined on homogeneous elements $f : A^{\otimes n} \to M$ by the formula

$$(df)(a_0,\dots,a_n) = a_0 f(a_1,\dots,a_n) + \sum_{i=1}^{n}(-1)^i f(a_0,\dots,a_{i-1}a_i,\dots,a_n) \qquad (2.3)$$
$$+ (-1)^{n+1} f(a_0,\dots,a_{n-1})a_n$$

It is easy to prove that $d^2 = 0$, and the corresponding cohomology is denoted by $H^\bullet(A,M)$, the *Hochschild cohomology* of A with values in M.

If $M = B$ is an algebra such that for any $a \in A$ and any $b, b' \in B$, $a(bb') = (ab)b'$ and $(bb')a = b(b'a)$ (B does not have to be a A algebra in the traditional sense, as we do not require that (the image of) A be central in B), then $(C^\bullet(A,B), d)$ becomes a DG algebra; the product $\cup$ is defined on homogeneous elements by

$$(f \cup g)(a_1,\dots,a_{m+n}) = f(a_1,\dots,a_m)g(a_{m+1},\dots,a_{m+n}) \qquad (2.4)$$

If $M = A$, then we write $HH^\bullet(A) := H^\bullet(A,A)$.

Unraveling the definition of the Hochschild differential, it is clear that the 0-th cohomology space $H^0(A,M)$ is equal to the space M^A of A-invariant elements in M (i.e. those elements on which the left and right actions coincide). In the case when $M = A$ is the algebra itself we then have $H^0(A,A) = Z(A)$, the center of A.

It is also clear that the Hochschild 1-cocycles are $\Bbbk$-linear maps $f : A \to M$ such that $f(ab) = af(b) + f(a)b$ for $a, b \in A$, i.e. derivations of A with values in M. The 1-coboundaries are derivations of the form $ad_m(a) := ma - am$ for $a \in A$

and $m \in M$. Thus $H^1(A, M)$ is the quotient of the space of derivations by the inner derivations.

Theorem 2.3.5 ([Ger64]). *$HH^2(A)$ is the set of infinitesimal deformations of A up to equivalence.*

Proof. Given an infinitesimal deformation $\star$ of A, the condition that $a \star b = ab$ mod ϵ means that for any $a, b \in A$, $a \star b = ab + \mu(a, b)\epsilon$, with $\mu : A \otimes A \to A$ a $\Bbbk$-bilinear map. The associativity of $\star$ is then equivalent to

$$a\mu(b, c) + \mu(a, bc) = \mu(a, b)c + \mu(ab, c)$$

which is easily seen to be exactly the condition that μ be a 2-cocycle, i.e. $d\mu = 0$. Conversely, any 2-cocycle allows us to define an infinitesimal deformation of A.

Two infinitesimal deformations $\star$ and $\star'$ are equivalent if there is an isomorphism of $k[\epsilon]/\epsilon^2$-algebras $(A[\epsilon]/\epsilon^2, \star) \xrightarrow{\sim} (A[\epsilon]/\epsilon, \star')$ which is the identity modulo ϵ. This last condition means that there exists a $\Bbbk$-linear map $f : A \to A$, such that the isomorphism maps a to $a + f(a)\epsilon$ and

$$\mu(a, b) + f(ab) = \mu'(a, b) + af(b) + f(a)b$$

which is equivalent to $\mu - \mu' = df$ and therefore $HH^2(A)$ is the set of infinitesimal deformations of A up to equivalences. □

Now, given an order n deformation of $\star$ of A, write $\star$

$$a \star b = ab + \sum_{i=1}^{n} \mu_i(a, b)\epsilon^i$$

where $\mu_i : A \otimes A \to A$ are $\Bbbk$-bilinear.

Proposition 2.3.6 ([Ger64]). *Let $\nu_{n+1} : A^{\otimes 3} \to A$ be the $\Bbbk$-trilinear map defined by*

$$\nu_{n+1}(a, b, c) := \sum_{i=1}^{n} (\mu_i(\mu_{n+1-i}(a, b), c) - \mu_i(a, \mu_{n+1-i}(b, c)))$$

Then the associativity of $\star$ up to order n is equivalent to ν_{n+1} being a 3-cocycle, i.e. $d\nu_{n+1} = 0$.

Given an order n deformation one can ask if it is possible to extend it to an order $n+1$ deformation. This means that we ask for a linear map $\mu_{n+1} : A \otimes A \to A$ such that

$$\sum_{i=0}^{n+1} \mu_i(\mu_{n+1-i}(a, b), c) = \sum_{i=0}^{n+1} \mu_i(a, \mu_{n+1-i}(b, c))$$

which is equivalent to solving the equation $d\mu_{n+1} = \nu_{n+1}$, where $d\nu_{n+1} = 0$. In other words, the only obstruction for extending deformations lies in $HH^3(A)$.

In particular, if $HH^3(A) = 0$, then we one can solve such equation for all n, and for each n, and the freedom in choosing solutions at each step, modulo equivalences, is the space $HH^2(A)$.

A *universal formal deformation* of A is a formal n-parameter deformation $\star_u$ such that, for every one-parameter formal deformation $\star$, there exists a unique continuous homomorphism $p : k[\![\epsilon_1, \ldots, \epsilon_n]\!] \to k[\![\epsilon]\!]$ such that $a \star b = p(a \star_u b)$. In other words, $\star$ is a specialization of $\star_u$.

We conclude this section with the following important result.

Proposition 2.3.7. *If $HH^1(A) = HH^3(A) = 0$, then there exists a universal formal deformation of A with base $k[\![HH^2(A)]\!]$.*

2.4 The rational Cherednik algebra

2.4.1 Equivariant differential operators

From now now, we assume that $\Bbbk = \mathbb{C}$. Let then V be a finite dimensional vector space over $\mathbb{C}$, $W \subset \mathrm{GL}(V)$ a finite group acting on V and let $\mathcal{D}(V) = \mathcal{D}(\mathbb{C}[V])$ be the algebra of differential operators on V. As W acts naturally on $\mathbb{C}[V] = \mathrm{Sym}\, V^*$, there is a natural action of W on $\mathcal{D}(V)$, such that for $w \in W$ and $D \in \mathcal{D}(V)$, we have

$$w \cdot D := wDw^{-1}$$

where the left and right $\mathbb{C}W$-module structures come from $\mathcal{D}(V) \subset \mathrm{End}(V)$. In other words, for $f \in \mathbb{C}[V]$, $(w \cdot D)(f) = w \cdot D(w^{-1} \cdot f)$.

A very natural question is that of identifying the subset $\mathcal{D}(V)^W$ of differential operators in $\mathcal{D}(V)$ that are invariant under W. First, notice that both $\mathbb{C}[V]$ and $\mathbb{C}[V^*]$ embed into $\mathcal{D}(V)$, where $\mathbb{C}[V]$ is realized by multiplication operators, and $\mathbb{C}[V^*] = \mathrm{Sym}\, V$ by constant coefficient differential operators. In fact, $\mathcal{D}(V) \simeq \mathbb{C}[V] \otimes \mathbb{C}[V^*]$ as vector spaces.

Passing to invariants, the previous discussion leads to realizing that the question of describing $\mathcal{D}(V)^W$ amounts to finding invariant elements in the tensor product $\mathbb{C}[V] \otimes \mathbb{C}[V^*]$. Obviously, this is bigger than simply $\mathbb{C}[V]^W \otimes \mathbb{C}[V^*]^W$. In fact, since $\mathbb{C}[V] \otimes \mathbb{C}[V^*] \simeq \mathbb{C}[V \oplus V^*]$, we now see that the original problem amounts to describing invariant elements in $\mathbb{C}[V \oplus V^*]$.

When W is a Coxeter group not of the type E6, E7, or E8, recall that the algebra $\mathbb{C}[V \oplus V^*]^W$ is generated as a *Poisson* algebra by $\mathbb{C}[V]^W$ and $\mathbb{C}[V^*]^W$. A simple argument involving passing to to associated graded then yields:

Theorem 2.4.1 ([Wal93], Theorem 2.2). *Under the condition above, $\mathcal{D}(V)^W$ is generated as an algebra by $\mathbb{C}[V]^W$ and $\mathbb{C}[V^*]^W$*

In fact, while the previous theorem is true, it is only partial and the condition on W is not a requirement for the conclusion, but only for the original proof. Indeed Levasseur and Stafford proved that in fact the previous result is true for *any* finite group.

Theorem 2.4.2 (Levasseur-Stafford [LS95], Theorem 5). *If $W \subset GL(V)$ is* any *finite group acting on a finite dimensional vector space V, then $D(V)^W$ is generated by its two subalgebras $\mathbb{C}[V]^W$ and $\mathbb{C}[V^*]^W$, where $\mathbb{C}[V]^W$ naturally embeds as the invariant constant coefficient differential operators.*

In looking for deformations of the ring of invariant differential operators one is led to considering not $\mathcal{D}(V)^W$ but a closely related ring. Indeed one can form the crossed product algebra $\mathcal{D}(V) \rtimes W$ and let

$$\boldsymbol{e} := \frac{1}{|W|} \sum_{w \in W} w \in \mathbb{C}W \subset \mathcal{D}(V) \rtimes W$$

then we have the following isomorphism:

$$D(V)^W \xrightarrow{\sim} \boldsymbol{e}\,(\mathcal{D}(V) \rtimes W)\,\boldsymbol{e}, \quad D \mapsto D\boldsymbol{e} = D\boldsymbol{e} = \boldsymbol{e}D\boldsymbol{e}$$

and thus we can deform $\mathcal{D}(V)^W$ by deforming $\mathcal{D}(V) \rtimes W$ first and then averaging. Note that this technique also worked in describing noncommutative deformations of Kleinian singularities in the works of W. Crawley-Boevey and M.P. Holland [CBH98]. In fact these two examples are special cases of a more general theory of symplectic reflection algebras, see [EG02b].

The following theorem provides the first step in deforming $\mathcal{D}(V) \rtimes W$.

Theorem 2.4.3 (Alev, Farinati, Lambre, Solotar [AFLS00]). *The odd cohomology of $\mathcal{D}(V) \rtimes W$ vanishes, and $HH^2(\mathcal{D}(V) \rtimes W)$ is the space $\mathbb{C}[S]^W$ of conjugation invariant functions on the set of reflections in W.*

It follows from theorem 2.4.3 and proposition 2.3.7 that there exists a universal deformation of $\mathcal{D}(V) \rtimes W$ parametrized by $\mathbb{C}[S]^W$. We recall in the following sections the construction of such universal deformations.

2.4.2 Dunkl operators

Let W be a finite reflection group, acting on its reflection representation V over $\mathbb{C}$, and let $\mathcal{A}$ be the set of all reflection hyperplanes. For any $H \in \mathcal{A}$, let $\alpha_H \in V^*$ define the reflection hyperplane H, i.e. $H = \ker \alpha_H$. We will write $\langle -, - \rangle$ for the natural pairing $V^* \times V \to \mathbb{C}$.

We think of V^* as linear functions on V, through the embedding $V^* \hookrightarrow \operatorname{Sym} V^* = \mathbb{C}[V]$, where $\mathbb{C}[V]$ is the algebra of regular functions on the affine space V. The evaluation of a function $z \in V^*$ at a point $\xi \in V$ is thus simply $z(\xi) = \langle z, \xi \rangle$. Note that we will usually reserve Roman letters for elements in V^* and Greek ones for elements in V.

Next, we define

$$\delta := \prod_{H\in\mathcal{A}} \alpha_H \in \mathbb{C}[V] \tag{2.5}$$

and let V_{reg} be the open subvariety $V \setminus \left(\bigcup_{H\in\mathcal{A}} H\right)$. Then $\mathbb{C}[V_{reg}] = \mathbb{C}[V][\delta^{-1}]$.

Both W and $\mathcal{D}(V_{reg})$ act on V_{reg}, and W acts naturally on $\mathcal{D}(V_{reg}) \subset \mathrm{End}_{\mathbb{C}}(\mathbb{C}[V_{reg}])$, and we can form the algebra $\mathcal{D}W := \mathcal{D}(V_{reg}) \rtimes W$.

Recall that given a reflection hyperplane H, W_H is the (pointwise) stabilizer of H, which is a cyclic subgroup of order $n_H \geq 2$, and finally, recall the definition of the idempotents associated to H:

$$e_{H,i} := \frac{1}{n_H} \sum_{w\in W_H} \det{}_V(w)^{-i} w \in \mathbb{C}W_H \subset \mathbb{C}W,\ i = 0, \dots, n_H - 1 \tag{2.6}$$

Let now $k := \{k_{H,i}\}_{H\in\mathcal{A}, i=0\dots n_H-1}$ be a set of parameters, where k is chosen to be constant on conjugacy classes of reflections. We also assume that $k_{H,0} = 0$ for all $H \in \mathcal{A}$, then define the element:

$$a_H = a_H(k) := \sum_{i=1}^{n_H-1} n_H k_{H,i} e_{H,i} \in \mathbb{C}W_H \tag{2.7}$$

We are ready for the main definition of this section (see [DO03]).

Definition 2.4.4 (Dunkl operators for complex reflection groups)**.** Let $\xi \in V$, then the Dunkl operator $T_\xi = T_\xi(k) \in \mathcal{D}W$ is defined by the formula

$$T_\xi(k) = \frac{\partial}{\partial\xi} - \sum_{H\in\mathcal{A}} \frac{\langle\alpha_H, \xi\rangle}{\alpha_H} a_H(k) \tag{2.8}$$

In the case when W is a finite Coxeter group, it is more convenient to label everything in terms of the subset of reflections $\Sigma \subset W$ rather than in terms of the reflection hyperplanes. The previous definitions take the following simpler form.

For a reflection $s \in \Sigma$, we have:

$$\begin{aligned}
H_s &= \ker \alpha_s \\
n_s &= 2 \\
W_{H_s} &= \{1, s\} \\
\det{}_V|_{H_s} &:= \text{sign} \\
e_{s,0} &= \frac{1}{2}(1+s) \\
e_{s,1} &= \frac{1}{2}(1-s) \\
a_s &= e_s k_{s,1} \epsilon_{s,1} = k_s(1-s) \\
\delta &:= \prod_{s \in \Sigma} \alpha_s
\end{aligned}$$

Let also $\alpha_s^\vee \in V$ be the (-1)-eigenvector of s such that $\langle \alpha_s, \alpha_s^\vee \rangle = 2$. Notice that for $x \in V^*$ and $\xi \in V$,

$$(1-s)x = \langle x, \alpha_s^\vee \rangle \alpha_s \tag{2.9}$$

$$(1-s)\xi = \langle \alpha_s, \xi \rangle \alpha_s^\vee \tag{2.10}$$

The definition of Dunkl operators thus becomes (see [Dun89]):

Definition 2.4.5 (Dunkl operators for Coxeter groups)**.** For $\xi \in V$, the Dunkl operator $T_\xi = T_\xi(k) \in \mathcal{D}W$ is defined by:

$$T_\xi(k) = \frac{\partial}{\partial \xi} - \sum_{s \in \Sigma} k_s \frac{\langle \alpha_s, \xi \rangle}{\alpha_s}(1-s) \tag{2.11}$$

Example 2.4.6. Let $W = \mathbb{Z}/2\mathbb{Z}$, $V = \mathbb{C}$. Then there is only one Dunkl operator up to scaling:

$$T = \frac{\partial}{\partial x} - \frac{k}{x}(1-s)$$

Note that in fact, T maps the space of polynomials $\mathbb{C}[x]$ to itself.

Before we move on to the main result about Dunkl operators, on which all their applications are based, we include the following easy proposition:

Proposition 2.4.7. *Let W be a finite Coxeter group, $\xi \in V$ and $x \in V^*$. Then*

(i) $[T_\xi, x] = \langle x, \xi\rangle - \sum_s k_s \langle \alpha_s, \xi\rangle \langle x, \alpha_s^\vee\rangle s$

(ii) If $w \in W$, then $wT_\xi w^{-1} = T_{w\xi}$. In particular, if $s \in W$ is a reflection, then $[s, T_\xi] = -\langle \alpha_s, \xi\rangle T_{\alpha_s^\vee} s$.

Proof. (i) The proof follows immediately from (2.9).

(ii) The first part is just a direct consequence of the invariance of k. For the second part, notice that by (2.10) $T_{s\xi} = T_{\xi - \langle \alpha_s, \xi\rangle \alpha_s^\vee} = T_\xi - \langle \alpha_s, \xi\rangle T_{\alpha_s^\vee}$, then compute $sT_\xi - T_\xi s = (sT_\xi s - T_\xi)s$.

□

We are now ready for the main result of this section.

Theorem 2.4.8 (Dunkl, [Dun89])**.** *The Dunkl operators commute:*

$$[T_{\xi_1}, T_{\xi_2}] = 0, \ \forall \xi_1, \xi_2 \in V$$

In the case when W is a finite Coxeter group, the proof of this theorem was originally found by Dunkl in [Dun89]. The general case is more technical and was proved by Dunkl and Opdam in [DO03].

Note that a consequence of the previous theorem is that it yields an embedding of algebras $\mathbb{C}[V^*] \hookrightarrow \mathcal{D}W$.

2.4.3 The rational Cherednik algebra

Using the notation from the previous section, we introduce the following definition:

Definition 2.4.9 ([DO03])**.** The *rational Cherednik algebra* $H_k(W)$ is the subalgebra of $\mathcal{D}W$ generated by $\mathbb{C}[V]$, $\mathbb{C}[V^*]$ and $\mathbb{C}W$. The subalgebras $\mathbb{C}[V]$ and $\mathbb{C}W$ are embedded in $\mathcal{D}W$ in the natural way and are independent of k. On the other hand, the embedding of $\mathbb{C}[V^*]$ in $\mathcal{D}W$ is defined via the Dunkl operators $T_\xi(k)$ which certainly depend on k.

It is also possible to give an abstract definition of Cherednik algebras in terms of generators and relations. From this point of view, the previous definition is called the *Dunkl representation* or the *Dunkl embedding*. The key point is that the Dunkl representation is faithful.

Theorem 2.4.10. *The algebra $H_k(W)$ is generated by the elements of V, V^* and W subject to the following relations*

$$[x, x'] = 0,\ \ [\xi, \xi'] = 0,\ \ wxw^{-1} = w(x),\ \ w\xi w^{-1} = w(\xi)$$

$$[\xi, x] = \langle \xi, x\rangle + \sum_{H\in\mathcal{A}} \frac{\langle \alpha_H, \xi\rangle\langle x, v_H\rangle}{\langle \alpha_H, v_H\rangle} \sum_{i=0}^{n_H-1} n_H(k_{H,i} - k_{H,i+1})e_{H,i}$$

where $x, x' \in V^*$, $\xi, \xi' \in V$, $w \in W$ and $v_H \in V$ is such that $\langle -, v_H\rangle = 0$ on H.

In the case when W is a finite Coxeter group, the previous relations take a simpler form. The rational Cherednik algebra $H_k(W)$ is the quotient of the algebra $T(V \oplus V^*) \rtimes W$ by the relations:

$$[x, x'] = 0,\ \ [\xi, \xi'] = 0,\ \ wxw^{-1} = w(x),\ \ w\xi w^{-1} = w(\xi)$$

$$[\xi, x] = \langle x, \xi\rangle - \sum_{s\in\Sigma} k_s\langle x, \alpha_s^\vee\rangle\langle \alpha_s, \xi\rangle s$$

where $x, x' \in V^*$, $\xi, \xi' \in V$ and $w \in W$

Note 2.4.11. Recall that we imposed the normalization condition $\langle \alpha_s, \alpha_s^\vee \rangle = 2$, which makes the relations slightly simpler.

Example 2.4.12. Consider $W = \mathbb{Z}/2$ acting on $V = \mathbb{C}\xi$ and $V^* = \mathbb{C}x$ where $\langle x, \xi \rangle = 1$. Then one can take $\alpha_s = x$ and $\alpha_s^\vee = 2\xi$. One then gets that $H_k(\mathbb{Z}/2)$ is generated by x, ξ and s satisfying:

$$s^2 = 1, \quad sx = -xs, \quad s\xi = -\xi s, \quad [\xi, x] = 1 - 2ks$$

Example 2.4.13. Let $W = S_n$ act on $\mathbb{C}^n$ by permuting the basis elements. All the reflections, corresponding the transpositions in S_n, are conjugate to one another and so here again there is only one parameter k. The algebra $H_k(S_n)$ is then isomorphic to the quotient of $\mathbb{C}\langle x_1, \dots, x_n, y_1, \dots, y_n \rangle \rtimes W$ by the relations

$$[x_i, x_j] = [y_i, y_j] = 0, \; [y_i, x_j] = ks_{ij}, \; [y_i, x_i] = \sum_{i \neq j} 1 - ks_{ij}$$

We will think of the Cherednik algebra as the algebra given the generators and relations as listed above; the Dunkl operators then provide an embedding

$$H_k(W) \hookrightarrow \mathcal{D}(V_{reg}) \rtimes W$$

We now list some of the main properties of the family of algbras $\{H_k\}_k$:

Theorem 2.4.14 ([EG02a][BEG03])**.** *Let $H_k = H_k(W)$ be the family of Cherednik algebras associated to a (complex) reflection group W.*

(i) $H_0 = \mathcal{D}(V) \rtimes W$, the crossed product of the Weyl algebra of V with W.

(ii) PBW property: the linear map $\mathbb{C}[V] \otimes \mathbb{C}W \otimes \mathbb{C}[V^*] \to H_k$ induced by multiplication in H_k is a $\mathbb{C}[V]$-module isomorphism.

In fact for both the standard filtration *on H_k (where $\deg V = \deg V^* = 1$, $\deg \mathbb{C}W = 0$) and the* differential filtration *(where $\deg V^* = \deg \mathbb{C}W = 0$, $\deg V = 1$), we have that $\mathrm{gr}(H_k) = \mathbb{C}[V \oplus V^*] \rtimes W$.*

(iii) The family $\{H_k\}$ represents the universal deformation of H_0.

(iv) Let $H_{reg} = H_k[\delta^{-1}]$ denote the localization of H_k at the Ore subset $\{\delta^k\}_{\{k\in\mathbb{N}\}}$, where $\delta = \prod_{s\in\Sigma} \alpha_s$. Then the induced map $H_{reg} \to \mathcal{D}(V_{reg}) \rtimes W$ is an isomorphism of algebras.

We focus now on the PBW property if H_k and we make restate in different way which will be very handy for doing computations later.

Lemma 2.4.15 (PBW Property for H_k)**.** *Let $\{\xi_i\}_{i=1\dots n}$ and $\{x_i\}_{i=1\dots n}$ be bases of V and V^* respectively. Suppose that*

$$\sum_{I=(i_1\dots i_n),J=(j_1\dots j_n)} \lambda_{I,J}\xi_1^{i_1}\cdots\xi_n^{i_n}x_1^{i_1}\cdots x_n^{i_n} = 0$$

inside H_k, where $\lambda_{I,J} \in \mathbb{C}W$. Then $\lambda_{I,J} = 0$ for all I and J.

Proof. This is just a restatement of the PBW theorem which gives that the elements $\xi_1^{i_1}\cdots\xi_n^{i_n}x_1^{i_1}\cdots x_n^{i_n}$ are linearly independent over $\mathbb{C}W$. □

2.4.4 The spherical subalgebra

Let us now introduce the idempotent

$$\boldsymbol{e} := \frac{1}{|W|}\sum_{w\in W} w \in \mathbb{C}[W].$$

Definition 2.4.16. The *spherical subalgebra* of H_k is the algebra $U_k := \boldsymbol{e}H_k\boldsymbol{e}$.

Notice that $1 \notin \boldsymbol{e}H_k\boldsymbol{e}$. On the other hand, since $\boldsymbol{e}x = x\boldsymbol{e} = x$ for $x \in \boldsymbol{e}H_k\boldsymbol{e}$, $\boldsymbol{e}$ is the unit for the spherical subalgebra. We can embed both $\mathbb{C}[V^*]^W$ and $C[V]^W$ into the spherical subalgebra as follows. Take $f \in C[V^*]^W$ (the other case is identical) and set $m_{\boldsymbol{e}}(f) = f\boldsymbol{e}$. Since f is invariant, we have $\boldsymbol{e}f\boldsymbol{e} = f\boldsymbol{e}^2 = f\boldsymbol{e} = m_{\boldsymbol{e}}(f)$, so that $m_{\boldsymbol{e}}$ actually maps $\mathbb{C}[V^*]^W$ to $\boldsymbol{e}H_k\boldsymbol{e}$. The injectivity is clear from the PBW-theorem. As for the fact that $m_{\boldsymbol{e}}$ is a homomorphism, we have $m_{\boldsymbol{e}}(fg) = fg\boldsymbol{e} = fg\boldsymbol{e}^2 = f\boldsymbol{e}g\boldsymbol{e} = m_{\boldsymbol{e}}(f)m_{\boldsymbol{e}}(g)$. From now on, we will consider both $\mathbb{C}[V^*]^W$ and $\mathbb{C}[V]^W$ as subalgebras of the spherical subalgebra.

Theorem 2.4.17 ([BEG03])**.** *$U_0 = \mathcal{D}(V)^W$ and the family $\{U_k\}$ is a universal deformation of $D(V)^W$. Also, we have $\operatorname{gr} U_k \simeq \mathbb{C}[V \oplus V^*]^W$.*

The Dunkl embedding $H_k(W) \hookrightarrow \mathcal{D}(V_{reg}) \rtimes W$ restricts to a map

$$U_k = \boldsymbol{e}H_k\boldsymbol{e} \to \boldsymbol{e}(DW)\boldsymbol{e} \xrightarrow{\sim} \mathcal{D}(V_{reg})^W$$

which is an injective homomorphism of unital algebras, called the spherical Dunkl representation of U_k. The localization isomorphism $H_k[\delta^{-1}] \xrightarrow{\sim} \mathcal{D}(V_{vreg}) \rtimes W$ restricts to an isomorphism

$$U_k[\delta^{-1}] \xrightarrow{\sim} \mathcal{D}(V_{reg})^W$$

The relationship between H_k and U_k depends drastically on the values of the parameter $k = \{k_H\}$. For values of the parameter k called *regular*, which include integral parameters, H_k is a simple algebra. The result on the Coxeter case is proved in [BEG03], while in general it follows from the semi-simplicity of a certain category $\mathcal{O}_{H_k}$ of H_k-modules, which can be found in [GGOR03].

A standard Morita theory argument [MR01, Proposition 3.5.6] shows that in fact

Theorem 2.4.18. *If k is* regular*, then H_k and $\boldsymbol{e}H_k\boldsymbol{e}$ are simple algebras, Morita equivalent to each other.*

In that situation, the following result holds:

Theorem 2.4.19 ([BEG03])**.** *If k is* regular *then the two subalgebras $\mathbb{C}[V]^W$ and $\mathbb{C}[V^*]^W$ generate the spherical subalgebra eH_ke.*

This theorem generalizes the previous theorem of Levasseur and Stafford which held for $\boldsymbol{e}H_0\boldsymbol{e} = \mathcal{D}(V)^W$.

2.5 Clifford Algebras

In this section we recall the definition and basic properties of the Clifford algebra on a vector space equipped with a symmetric bilinear form. We then study the special case where the vector space is of the form $V \oplus V^*$ equipped with the canonical bilinear form.

Throughout this section, we assume that $\mathrm{char}(\Bbbk) \neq 2$. We then have the following definition:

Definition 2.5.1 (Clifford algebra)**.** Let V be a finite-dimensional $\Bbbk$-vector space together with a symmetric bilinear form, $\phi : V \times V \to \Bbbk$, and associated quadratic form, $q(v) := \phi(v, v)$. The *Clifford algebra* associated with V and ϕ (or equivalently q) is an associative $\Bbbk$-algebra, $Cl(V, \phi)$, together with a linear map, $i_\phi : V \to Cl(V, \phi)$, satisfying the condition

$$i_\phi(u)i_\phi(v) + i_\phi(v)i_\phi(u) = 2\phi(u, v) \ \text{ for all } u, v \in V \tag{2.12}$$

or equivalently

$$i_\phi(v)^2 = q(v) \ \text{ for all } v \in V \tag{2.13}$$

and which is universal for that property.

In explicit terms, the universal property says that for every algebra A and every linear map $f : V \to A$, with $f(v)^2 = q(v)$ for all $v \in V$, there exists an algebra homomorphism $\overline{f} : Cl(V) \to A$, such that $f = \overline{f} i_\phi$.

The uniqueness of the Clifford algebra, follows by a standard universality argument. As for the existence, it is easy to show that the following algebra indeed satisfies the required universal property:

$$Cl(V, \phi) \simeq TV/ \langle u \otimes v + v \otimes u - 2\phi(u, v),\ u, v \in V\rangle \tag{2.14}$$

where TV is the tensor algebra of V over $\Bbbk$. Notice that if $\phi = 0$, then $Cl(V, 0) = \bigwedge V$.

We will need the following two standard results:

Proposition 2.5.2. *If* $\dim V = n$, *then* $\dim Cl(V) = 2^n$. *If* $x_1, \dots, x_n$ *is an orthogonal basis in* V, *then the set* $\{x_{i_1} \dots x_{i_s} : i_1 < \dots < i_s\}$ *is a basis of* $Cl(V)$.

Theorem 2.5.3 ([Gre67]). *Let* $\Bbbk$ *be algebraically closed and assume that* ϕ *is a non-degenerate symmetric bilinear form on* V. *If* $\dim V = 2m$, *then* $Cl(V, \phi) \simeq M_{2^m}(\Bbbk)$, *and if* $\dim V = 2m + 1$, *then* $Cl(V, \phi) \simeq M_{2^m}(\Bbbk) \oplus M_{2^m}(\Bbbk)$.

Let us consider the following special case. Let V a finite dimensional vector space and let V^* be its dual. Then $V \oplus V^*$ has a canonical symmetric bilinear form $\langle , \rangle_{Cl}$. Explicitly, for $\xi, \eta \in V$ and $x, y \in V^*$, we have:

$$\langle \xi + x, \eta + y\rangle_{Cl} := \frac{1}{2} \left(\langle x, \eta\rangle + \langle y, \xi\rangle\right)$$

where again, $\langle , \rangle$ is the duality $V^* \times V \mapsto \mathbb{C}$. In particular,

$$\langle \xi, \eta \rangle_{Cl} = 0$$
$$\langle x, y \rangle_{Cl} = 0$$
$$\langle \xi, x \rangle_{Cl} = \frac{1}{2} \langle x, \xi \rangle$$

Proposition 2.5.4. *The Clifford algebra $Cl(V \oplus V^*)$ of $V \oplus V^*$ with the canonical bilinear form is the quotient of the tensor algebra $T(V \oplus V^*)$ by the ideal generated by the relations*

$$\xi\eta = -\eta\xi \qquad (\textit{and so } \xi^2 = 0)$$
$$xy = -yx \qquad (\textit{and so } x^2 = 0)$$
$$\xi x + x\xi = \langle x, \xi \rangle$$

where $\xi \in V$ and $x \in V^*$. Notice that there are natural embeddings:

$$\bigwedge V \hookrightarrow Cl(V \oplus V^*)$$
$$\bigwedge V^* \hookrightarrow Cl(V \oplus V^*)$$

where $\bigwedge V$ and $\bigwedge V^*$ are the exterior algebras of V and V^* respectively.

Let $\xi_1, \ldots, \xi_n$ and $x_1, \ldots, x_n$ be dual bases of V and V^*, where $n = \dim V$. Let us assume that $\Bbbk$ is algebraically closed and let $\imath \in \Bbbk$ be such that $\imath^2 = -1$. Next define the following elements of $Cl(V \oplus V^*)$

$$e_k^+ := \xi_k + x_k, \quad e_k^- := \imath(\xi_k - x_k) \tag{2.15}$$

for $k = 1, \ldots, n$. Then the e_k^+ and $e_k^-, k = 1, \ldots, n$ form a basis of $V \oplus V^*$ and we have the following relations inside $Cl(V \oplus V^*)$:

$$e_k^{\pm 2} = 1, \quad e_k^\pm e_l^\pm = -e_l^\pm e_k^\pm \text{ for } k \neq l$$

We now conclude this section by providing an explicit representation of $Cl(V \oplus V^*)$ as a matrix algebra when $n := \dim V = 1, 2$.

(n = 1) Let $V = \mathbb{C}\xi$ and $V^* = \mathbb{C}x$ s.t. $\langle x, \xi \rangle = 1$. Then $Cl(V \oplus V^*)$ is isomorphic to $M_2(\mathbb{C})$ via the map

$$\xi \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$x \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

(n = 2) Let $V = \mathbb{C}\xi_1 \oplus \mathbb{C}\xi_2$ and $V^* = \mathbb{C}x_1 + \mathbb{C}x_2$ s.t. $\langle x_i, \xi_j \rangle = \delta_{ij}$. Then we know $Cl(V \oplus V^*)$ is isomorphic to $M_4(\mathbb{C})$. Let $e_1^\pm$ and $e_2^\pm$ be defined as above.

It is well known that the Weyl represention of the Dirac matrices provide a representation of $e_1^\pm, e_2^\pm$. Explicitly:

$$e_1^+ \mapsto \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \; e_1^- \mapsto \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}$$

$$e_2^+ \mapsto \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \; e_2^- \mapsto \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}$$

Using the identities in equation (2.15), we finally obtain the representation we are looking for:

$$\xi_1 \mapsto \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \; x_1 \mapsto \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\xi_2 \mapsto \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \; x_2 \mapsto \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

CHAPTER 3
EQUIVARIANT DIFFERENTIAL OPERATORS ON DIFFERENTIAL FORMS

We now move on to our study of differential forms and their rings of differential operators. We provide here a self contained introduction to the theory of differential operators on graded commutative rings and we use this machinery to study the algebra of differential operators on forms first, and then the algebra of equivariant operators in order to prove the theorem stated in the introduction. We start with a brief review of graded algebra.

3.1 Graded algebra

Recall that a $\mathbb{Z}$-graded vector space is a vector space V together with a collection of subspaces $\{V^i\}_{i\in\mathbb{Z}}$ indexed by $i \in \mathbb{Z}$ such that $V = \bigoplus_{i\in\mathbb{Z}} V^i$. For $v \in V^i \subset V$, we write $|v| := i$ and say that V is homogeneous of degree i. The dual of such a vector space is the graded vector space V^* such that $(V^*)^i = (V^{-i})^*$.

Graded vector spaces form a category and we describe the maps. For U and V two graded vector spaces let $\underline{\mathrm{Hom}}_{\Bbbk}(U,V) := \bigoplus_i \underline{\mathrm{Hom}}^i_{\Bbbk}(U,V)$ as a graded vector space, where

$$\underline{\mathrm{Hom}}^i_{\Bbbk}(U,V) = \left\{f \in \mathrm{Hom}_{\Bbbk}(U,V) : f(U^n) \subset V^{n+i},\ \text{for all } n\right\}$$

The elements in $\underline{\mathrm{Hom}}^i_{\Bbbk}(U,V)$ are called linear maps of degree i, and a morphism $f : U \to V$ is simply a linear map of degree 0.

There exists a *shift functor* on the category of graded vector spaces, where the shift of the graded vector space V by n is the graded vector space $V[n]$ defined by

$(V[n])^i := V^{i+n}$, i.e. V is shifted to the *right*.

The tensor product of two graded vector spaces is still a graded vector space, with grading defined by

$$(U \otimes V)^n = \bigoplus_{i+j=n} U^i \otimes V^j$$

This operation can be repeated and in particular, tensoring V with itself k times, one has:

$$(V \otimes \cdots \otimes V)^n = \bigoplus_{i_1+\cdots i_k=n} V^{i_1} \otimes \cdots \otimes V^{i_k}$$

Similar to the notion of graded vector spaces and graded morphisms, there is a notion of graded rings and algebras. A *graded algebra* is a graded vector space $A = \bigoplus_i A^i$ such that the multiplication on A satisfies:

$$A^n \cdot A^m \subset A^{n+m} \quad \text{for all } n, m$$

Note that $A^0 \subset A$ is a subalgebra. One similarly defines the notion of a morphism between graded algebras.

A graded algebra A is said to be *graded commutative* if for homogeneous elements u, and v

$$uv = (-1)^{|u||v|} vu$$

Given two graded algebras A and B, the tensor product $A \otimes B$ is a graded algebra for the product defined on homogenous elements by:

$$(a \otimes b) \cdot (a' \otimes b') := (-1)^{|b||a'|} aa' \otimes bb'$$

Notice that this operation preserves graded commutativity as can easily be checked.

There are two natural ways to build a graded algebra starting with a graded vector space. First, notice that the standard tensor algebra $TV = \bigoplus_k V^{\otimes k}$ on a graded vector space V inherits a structure of graded algebra as follows:

$$
\begin{aligned}
(TV)^n &= \bigoplus_{k\geq 0} (V^{\otimes k})^n = \bigoplus_{k\geq 0} \bigoplus_{i_1+\cdots i_k=n} V^{i_1}\otimes\cdots\otimes V^{i_k} \\
&= \operatorname{span}\{v_1\otimes\cdots\otimes v_k,\ v_i \text{ homogeneous s.t. } |v_1|+\cdots+|v_k|=n\}
\end{aligned}
$$

We the have the following definition:

Definition 3.1.1. The *graded symmetric* algebra $\underline{Sym}\,V$ on the graded vector space V is the quotient algebra

$$\underline{Sym}\,V := TV/\left\langle u\otimes v-(-1)^{ij}v\otimes u,\ u\in V^i, v\in V^j\right\rangle$$

The grading on TV is preserved when passing to the quotient by a homogeneous ideal, and $\underline{Sym}\,V$ inherits a natural grading.

By definition, $\underline{Sym}\,V$ is a graded commutative algebra, and is in fact a free object in the category of graded commutative algebras. Any morphism from $\underline{Sym}\,V$ to another graded commutative algebra is completely determined by the images of elements in V. This is the universal property of $\underline{Sym}\,V$.

Finally, a graded module over a graded algebra A is a graded vector space M which is also an A-module where the action of A on M satisfies:

$$A_n\cdot M_m \subset M_{n+m}$$

and the notion of morphisms between graded modules is the natural one.

3.2 Polyvectors and differential operators.

First, recall that a *Gerstenhaber algebra* is a graded space $\mathcal{V}^\bullet$ together with

(i) A graded commutative associative algebra structure on $\mathcal{V}^\bullet$.

(ii) A graded (super) Lie algebra structure on $\mathcal{V}^\bullet[1]$ such that

$$[a, bc] = [a, b]c + (-1)^{(|(|a)-1)|(|b)} b[a, c]$$

Example 3.2.1. Let $\mathfrak{g}$ be a Lie algebra. Then $\bigwedge^\bullet \mathfrak{g}$ is a Gerstenhaber algebra. The product is the exterior product, and the bracket is the unique bracket which turns $\bigwedge^\bullet \mathfrak{g}$ into a Gerstenhaber algebra and which is the Lie bracket on $\mathfrak{g} = \bigwedge^1 \mathfrak{g}$.

Example 3.2.2. Let M be a smooth manifold. Then the algebra of polyvector fields $\mathcal{V}^\bullet_M = \bigwedge^\bullet T_M$ is a sheaf of Gerstenhaber algebra. The product is the exterior product, and the bracket is the Schouten bracket. We denote by $\mathcal{V}^\bullet(M)$ the Gerstenhaber algebra of global sections of this sheaf. The previous example is the algebra of left-invariant polyvector fields on the Lie group of $\mathfrak{g}$.

Given a Gerstenhaber algebra, one can construct an associative graded algebra whose definition is reminiscent of that of the universal enveloping algebra of a Lie algebra. More precisely, if $\mathcal{V}^\bullet$ is a Gerstenhaber algebra, the *enveloping algebra* of $\mathcal{V}^\bullet$ is defined to be the associative algebra $\mathcal{U}(\mathcal{V}^\bullet)$ generated by two sets of generators $i_a, L_a, a \in \mathcal{V}$, both linear in a, with degrees

$$|i_a| = |a|; \quad |L_a| = |a| - 1$$

and subject to the relations

$$i_{ab} = i_a i_b; \quad [L_a, L_b] = L_{[a,b]}$$

$$[L_a, i_b] = i_{[a,b]}; \quad L_{ab} = L_a i_b + (-1)^{|a|} i_a L_b$$

The algebra $\mathcal{U}(\mathcal{V}^\bullet)$ is equipped with the differential d of degree one which is defined as a derivation sending i_a to L_a and L_a to zero.

For a smooth manifold M one has an obvious homomorphism $\mathcal{U}(\mathcal{V}^\bullet(M)) \to \mathcal{D}(\Omega^\bullet(M))$, where the right hand side is the algebra of differential operators on differential forms on M. Tamarkin and Tsygan [TT05] claim that this map is in fact an isomorphism. We clarify these statements in the algebraic case when M is a smooth affine variety, and ΩM is an algebraic de Rham complex of M.

3.3 Differential operators on graded commutative rings

In this section we provide a self-contained introduction to the theory of differential operators on graded commutative rings. The definition parallels the Grothendieck one in the classical case and one simply replaces standard commutators with graded ones to keep the machinery working. For simplicity, we will work over $\Bbbk = \mathbb{C}$.

Recall that if A is a graded commutative $\mathbb{C}$-algebra, and P and Q are two graded left A-modules, the graded vector space $\underline{\mathrm{Hom}}_{\mathbb{C}}(P, Q) := \bigoplus_{k\in\mathbb{Z}} \underline{\mathrm{Hom}}^k_{\mathbb{C}}(P, Q)$ of $\mathbb{C}$-linear homomorphisms $\phi : P \to Q$ can be equipped with both a left and a right graded A-module structure:

$$(a\phi)(p) := a\phi(p), \qquad (\phi a)(p) := \phi(ap) \qquad a \in A, p \in P$$

One can then define for $a \in A$, the operator ad_a, which when applied to $\phi : P \to Q$, yields:

$$\mathrm{ad}_a \phi := a\phi - (-1)^{|a||\phi|}\phi a$$

In fact, for $X \in \underline{\mathrm{Hom}}^k_{\mathbb{C}}(A, A)$ and $Y \in \underline{\mathrm{Hom}}^l_{\mathbb{C}}(A, A)$, one defines the *graded commutator* of X and Y by the following:

$$[X, Y] := XY - (-1)^{kl} YX$$

With this notation, we have for $a \in A^k \subset \underline{\mathrm{Hom}}^k_{\mathbb{C}}(A, A)$ and $X \in \underline{\mathrm{Hom}}^l_{\mathbb{C}}(A, A)$

$$\mathrm{ad}_a X = [a, X]$$

We are now ready for the following definition, which looks identical to the one in the classical case:

Definition 3.3.1. An element $D \in \underline{\mathrm{Hom}}_{\mathbb{C}}(P, Q)$ is called a *Q-valued differential operator on P* of *order* at most k, if

$$\mathrm{ad}_{a_0} \mathrm{ad}_{a_1} \cdots \mathrm{ad}_{a_k} D = 0, \ \forall a_0, \cdots, a_k \in A$$

We write $\mathcal{D}^k(P, Q)$ for the set of such morphisms, in other words we define:

$$\mathcal{D}^k(P, Q) := \{D \in \underline{\mathrm{Hom}}_{\mathbb{C}}(P, Q) : \mathrm{ad}_{a_0} \mathrm{ad}_{a_1} \cdots \mathrm{ad}_{a_k} D = 0 \ \forall a_0, \cdots, a_k \in A\}$$

The ring of Q-valued differential operators on P is then defined to be the filtered ring

$$\mathcal{D}(P, Q) = \bigcup_{k \geq 0} \mathcal{D}^k(P, Q)$$

Theorem 3.3.2. *Suppose $P = Q = A$ is a graded commutative ring, then one has the following:*

$$\begin{aligned} \mathcal{D}^0(A) &= A \quad \textit{by left multiplication} \\ \mathcal{D}^1(A) &= A \oplus \underline{\mathrm{Der}}(A) \end{aligned}$$

where $\underline{\mathrm{Der}}(A)$ is the graded Lie algebra of graded derivations of A, *i.e.*

$$\underline{\mathrm{Der}}(A) = \left\{ \delta \in \underline{\mathrm{Hom}}_{\mathbb{C}}(A, A) : \delta(ab) = \delta(a)b + (-1)^{|a||\delta|} a\delta(b) \right\}$$

and the Lie bracket is simply the graded commutator *coming from* $\underline{\mathrm{Hom}}$.

Proof. The proof of this statement is virtually identical to the one in the classical case when A is commutative. The definition of ad_a uses the graded commutator and thus it is clear that $\mathcal{D}^0 \xrightarrow{\sim} A$, where the isomorphism is $\phi \mapsto \phi(1)$ for $\phi \in \mathcal{D}^0(A)$.

If $\phi \in \mathcal{D}^1$, then $\operatorname{ad}_a \phi \in \mathcal{D}^0 = A$, and thus for all $a \in A$, there exists $\lambda(a) \in A$ such that

$$(\operatorname{ad}_a \phi)(b) = a\phi(b) - (-1)^{|a||\phi|}\phi(ab) = \lambda(a)b \tag{3.1}$$

Specializing (3.1) at $b = 1$ then yields

$$\lambda(a) = a\phi(1) - (-1)^{|a||\phi|}\phi(a)$$

which, when plugged back into (3.1), gives

$$\begin{aligned} \phi(ab) &= (-1)^{|a||\phi|}a\phi(b) - (-1)^{|a||\phi|}a\phi(1)b + \phi(a)b \\ &= \phi(a)b + (-1)^{|a||\phi|}a\phi(b) + \phi(1)ab \end{aligned} \tag{3.2}$$

and thus

$$\phi = (\phi - \phi(1)\cdot) + \phi(1)\cdot$$

where $\phi(1)\cdot$ is the multiplication operator by $\phi(1) \in A$, and $(\phi - \phi(1)\cdot)$ is a graded derivation as can be seen from (3.2).

Finally, the sum $A \oplus \underline{\operatorname{Der}}(A)$ is indeed a direct sum, because if $\phi \in A \cap \underline{\operatorname{Der}}(A)$, then for $a, b \in A$, $\phi(ab) = \phi(1)ab$ and thus

$$\begin{aligned} \phi(1)ab &= \phi(a)b + (-1)^{|a||\phi|}a\phi(b) \\ &= \phi(1)ab + (-1)^{|a||\phi|}a\phi(1)b \\ &= \phi(1)ab + (-1)^{|a||\phi|}(-1)^{|a||\phi|}\phi(1)ab \\ &= 2\phi(1)ab \end{aligned}$$

and taking $a = b = 1$ shows that $\phi(1) = 0$, and hence $\phi = 0$. □

Note 3.3.3. When A is graded commutative, $\underline{\mathrm{Der}}(A)$ is naturally a graded left A-module.

We also remind the reader about the following identities involving graded commutators of graded morphisms:

$$[a, bc] = [a, b]c + (-1)^{|a||b|}b[a, c]$$
$$[ab, c] = a[b, c] + (-1)^{|b||c|}[a, c]b$$
$$[a, [b, c]] = [[a, b], c] + (-1)^{|a||b|}[b, [a, c]]$$
$$[[a, b], c] = [a, [b, c]] + (-1)^{|b||c|}[[a, c], b]$$

3.4 Free graded commutative algebras

As we have mentioned before, it is well known that in general, $\mathcal{D}^1(A)$ doesn't generate $\mathcal{D}(A)$. In the case where A is a nonsingular commutative ring though, it is also well known that it does. We will now see that this result holds in the case of free graded commutative algebras. The proofs in this section are adaptation to the graded commutative setting of standard arguments that show the simplicity of the Weyl algebra in n variables.

From now on till the end of this section, let R be a free graded commutative algebra, generated by $\xi_1, \ldots, \xi_n$, with degrees $|\xi_1|, \ldots, |\xi_n|$, and let ∂_i be the graded derivation of R of degree $-|\xi_i|$, defined by $\partial_i(\xi_j) = \delta_i^j$.

Lemma 3.4.1. *The derivations ∂_i are well defined.*

Proof. R is the quotient of the free algebra on $\xi_1, \dots, \xi_n$, by the relations $\xi_j\xi_k = (-1)^{|\xi_j||\xi_k|}\xi_k\xi_j$ for all j, k. To check that ∂_i is well defined, all we have to do is check that

$$\partial_i(\xi_j\xi_k) = (-1)^{|\xi_j||\xi_k|}\partial_i(\xi_k\xi_j)$$

Clearly, we only need to check the previous equality when $i = j$ for example. In that case, the left hand side is simply ξ_k, while the right hand side is

$$(-1)^{|\xi_i||\xi_k|}\partial_i(\xi_k)\xi_i + (-1)^{|\xi_i||\xi_k|}(-1)^{|\xi_k||\partial_i|}\xi_k\partial_i(\xi_i) = (-1)^{|\xi_k|(|\xi_i|-|\xi_i|}\xi_k = \xi_k$$

□

In fact, the derivations ∂_i generate $\underline{\mathrm{Der}}\, R$ as the following lemma shows.

Lemma 3.4.2. *(i) For $\theta \in \underline{\mathrm{Der}}\, R$*

$$\theta = \sum_{i=1}^{n} [\theta, \xi_i]\partial_i$$

(ii) $\underline{\mathrm{Der}}\, R$ is freely generated as an R-module by the derivations $\partial_i, i = 1 \dots n$.

Proof. (i) Let $\theta' := \sum_{i=1}^{n}[\theta, \xi_i]\partial_i$. Clearly, θ' is a graded derivation of R. The action of θ' on ξ_j is

$$\left[\sum_{i=1}^{n}[\theta, \xi_i]\partial_i, \xi_j\right] = \sum_{i=1}^{n}[\theta, \xi_i][\partial_i, \xi_j] = [\theta, \xi_j]$$

Hence θ and θ' are two derivations of R which agree on generators of R, and thus $\theta = \theta'$.

(ii) The proof of (i) shows that $\underline{\mathrm{Der}}\, R$ is generated by the ∂_i. To show that they are linearly independent over R, suppose we have an equation

$$\sum_{i=1}^{n} r_i\partial_i = 0$$

where $r_i \in R$. Then computing $[\sum_{i=1}^n r_i \partial_i, \xi_j]$ for $j = 1 \dots n$ shows that $r_j = 0$ for all j.

$\square$

In order to study the sub-algebra of $\mathcal{D}(R)$ generated by $\mathcal{D}^1(R)$, we start with the following definition:

Definition 3.4.3. Let $\Delta^r = \Delta^r(R)$ be the subset of $\mathcal{D}(R)$ of differential operators on R generated by products of $s \leq r$ (graded) derivations of R, and $\Delta = \Delta(R) = \bigcup_{r \geq 0} \Delta^r(R)$.

In order to better understand $\Delta(R)$, we prove the following few technical facts which the generalize the familiar commutative setup.

Lemma 3.4.4. *(i) If ξ_i is of odd degree, then $\partial_i^n = 0$ for $n \geq 2$.*

(ii) Every element $P \in \Delta^r$ can be written as a linear combination of the form

$$\sum_{\beta_1 + \dots + \beta_n \leq r} p_\beta \partial^\beta$$

where $p_\beta := p_{\beta_1 \dots \beta_n} \in R$, $\partial_\beta := \partial_1^{\beta_1} \cdots \partial_n^{\beta_n}$, and $p_\beta = 0$ if $\beta_j \geq 2$ for some $j \in \{1, \dots, n\}$, and ξ_j is of odd degree. We call such a linear combination a standard form *for P.*

(iii) If ξ_i is of even degree then

$$\left[\partial_1^{\beta_1} \cdots \partial_n^{\beta_n}, \xi_i\right] = \beta_i \partial_1^{\beta_1} \cdots \partial_i^{\beta_i - 1} \cdots \partial_n^{\beta_n} \tag{3.3}$$

If ξ_i is of odd degree and $\beta_i \in \{0, 1\}$ then

$$\left[\partial_1^{\beta_1} \cdots \partial_n^{\beta_n}, \xi_i\right] = \varepsilon(i, \beta) \beta_i \partial_1^{\beta_1} \cdots \partial_i^{\beta_i - 1} \cdots \partial_n^{\beta_n} \tag{3.4}$$

where $\partial_i^{-1} := 0$ *and*

$$\varepsilon(i,\beta) := (-1)^{-|\xi_i|\left[|\xi_{i+1}||\beta_{i+1}|+\cdots+|\xi_n||\beta_n|\right]}$$

(iv) *Let* $\sum_{\beta_1+\cdots\beta_n\leq r} p_\beta \partial^\beta$ *be a standard form for* $P \in \Delta^r$, *and let* $\beta^* = (\beta_1^*, \ldots, \beta_n^*)$ *be such that* $\beta_1^* + \cdots + \beta_n^* = r$. *Define* $\mathrm{ad}_\xi := [\xi, -]$ *and* ad_ξ^m *to be the composition of* ad_ξ *m-times. Then*

$$\mathrm{ad}_{\xi_n}^{\beta_n^*} \cdots \mathrm{ad}_{\xi_1}^{\beta_1^*} P = \lambda(\beta_1^* \ldots \beta_n^*) p_{\beta_1^* \ldots \beta_n^*} \tag{3.5}$$

for some non-zero integer $\lambda(\beta_1^* \ldots \beta_n^*)$. *Hence, if* p_β *and* p'_β *are two standard forms for* P, $p_\beta = p'_\beta$ *for all* β.

(v) *If* $P \in \Delta^r$ *is such that* $[P, \xi_i] = 0$ *for some* $i \in \{1, \ldots n\}$, *then the standard form of* P *has* $p_\beta = 0$ *whenever* $\beta_i > 0$.

(vi) *If* $Q \in \Delta^r$ *and* ξ_i *is of odd degree, then the standard form of* $P := [Q, \xi_i]$ *has* $p_\beta = 0$ *whenever* $\beta_i > 0$.

Proof. (i) Since R is a free graded commutative algebra, every element in R can be written as a sum $\sum_{\alpha_1\cdots\alpha_n} \lambda_{\alpha_1\cdots\alpha_n} \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}$, where $\lambda_{\alpha_1\cdots\alpha_n} \in \mathbb{C}$, and $\lambda_{\alpha_1\cdots\alpha_n} = 0$ if $\alpha_j \geq 2$ for some j and ξ_j is of odd degree (because then $\xi_j^2 = 0$). It is clear that applying ∂_i (at least) twice to such an element would yield zero, and thus $\partial_i^n = 0$ for $n \geq 2$.

(ii) First, notice that the graded commutator $[\partial_i, \partial_j] = 0$, as it is a derivation and hence we can check equality on generators. Next, assume WLOG that $P = \theta_1 \cdots \theta_r$, where $\theta_i \in \underline{\mathrm{Der}}\, R$. Then expand each θ_i in terms of the ∂_j's, and use commutators to move all the coefficients in R to the left, and rearrange all the ∂_j's in order. Notice that by doing so, one can only reduce the total number of ∂_j's present in the original expression for P.

(iii) First, it is clear that the equality

$$\left[\partial_1^{\beta_1}\cdots\partial_n^{\beta_n},\xi_i\right] = \varepsilon(i,\beta)\partial_1^{\beta_1}\cdots[\partial_i^{\beta_i},\xi_i]\cdots\partial_n^{\beta_n}$$

holds, whether ξ_i is odd or even. Next, compute

$$\left[\partial_i^{\beta_i},\xi_i\right] = \partial_i^{\beta_i-1}[\partial_i,\xi_i] + (-1)^{|\xi_i||\partial_i|}\left[\partial_i^{\beta_i-1},\xi_i\right]\partial_i$$

If ξ_i is even, then so is $|\xi_i||\partial_i| = -|\xi_i|^2$, and thus

$$\left[\partial_i^{\beta_i},\xi_i\right] = \partial_i^{\beta_i-1}[\partial_i,\xi_i] + \left[\partial_i^{\beta_i-1},\xi_i\right]\partial_i$$

and the formula follows by induction.

If ξ_i is of odd degree and $\beta_i \in \{0,1\}$, then two side of equation (3.4) still agree, given the convention $\partial_i^{-1} = 0$. In fact, the only time the two side disagree is when $\beta_i = 2$, in which case the left hand side is zero, while the right hand side isn't necessarily.

(iv) Repeated use of (iii) shows that all the terms in the sum for which $\beta_i < \beta_i^*$ vanish. Hence, only the terms for which $\beta_i \geq \beta_i^*$ and $\beta_1 + \cdots + \beta_n = r = \beta_1^* + \cdots + \beta_n^*$ remain. This implies that the only term left is of the form $\pm(\beta_1^*!)\cdots(\beta_n^*!)p_{\beta_1^*\ldots\beta_n^*}$.

(v) Starting with a standard form for P, use (iii) to obtain a standard form for $[P,\xi_i]$, and then apply (iv) inductively on the homogeneous terms of P, starting with those with highest total degree $\beta_1 + \cdots + \beta_n$.

(vi) Follows directly from (ii) and (iii).

$\square$

The main result in this section is a generalization of the standard multivariable calculus argument which states that if $f_1,\ldots,f_n$ are smooth functions on a simply connected domain such that $\frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i}$ for all i, j, then there exists a function g such that $\frac{\partial g}{\partial x_i} = f_i$ for all i. Equivalently, if $d(f_i dx_1 + \cdots f_n dx_n) = 0$, i.e. it is a closed form, then there exists a function g such that $dg = f_i dx_1 + \cdots f_n dx_n$, i.e. $f_i dx_1 + \cdots f_n dx_n$ is in fact exact.

The standard calculus approach to find such a g, is to integrate each f_i with respect to x_i and then differentiate back to match the constants. Acting on Δ^r, we have seen that ad_{ξ_i} behaves like a derivative on the ∂_j's. The existence of odd variables in R make integration a lot more subtle, as can be seen in the following example.

Example 3.4.5. Let $R = \underline{Sym}\,\mathbb{C}\xi$, where $\deg \xi = 1$. Then $\underline{\mathrm{Der}}(R)$ is generated over R by ∂, and so Δ^1 is generated by ξ and ∂ with relations

$$\xi^2 = \partial^2 = 0,\ \partial\xi + \xi\partial = 1$$

Then $[\partial\xi, \xi] = \xi$ and hence ξ has an antiderivative in $\Delta^1 \subset \Delta^2$. On the other hand, there is no $Q \in \Delta^2$ such that $[Q, \xi] = \partial$. The problem here is that the only candidate would be ∂^2, but ∂^2 is zero and in fact $\Delta^2 = \Delta^1$.

On the other hand, if $D \in \mathcal{D}(R)$, then

$$[\xi, [\xi, D]] = [[\xi, \xi], D] + (-1)^{|\xi||\xi|}[\xi, [\xi, D]] = -[\xi, [\xi, D]]$$

and hence $[\xi, [\xi, D]] = 0$. Since D commutes with constants, this shows that $D \in \mathcal{D}^1$. Hence, $\mathcal{D} = \mathcal{D}^1 = \Delta^1$ and $\mathcal{D} = \Delta$.

The previous example motivates the following definition:

Definition 3.4.6. Let $P \in \mathcal{D}$. We will say that P is ξ_i*-integrable* if either ξ_i is even, or ξ_i is odd and $[P, \xi_i] = 0$.

Remark 3.4.7. If ξ_i is odd and if $P \in \Delta^r$ is ξ_i-integrable, then the standard form for P has all $p_{\beta_1,\ldots,\beta_n} = 0$ whenever $\beta_i > 0$. Also, for any $P \in \mathcal{D}$, $[P, \xi_i]$ is always ξ_i-integrable, since as in the example above,

$$\begin{aligned}
[[P, \xi_i], \xi_i] &= [P, [\xi_i, \xi_i]] + (-1)^{|\xi_i|^2}[[P, \xi_i], \xi_i] \\
&= -[[P, \xi_i], \xi_i]
\end{aligned}$$

Lemma 3.4.8. *We have* $\Delta^r = \Delta^{r+1} \cap \mathcal{D}^r(R)$.

Proof. Clearly, $\Delta^r \subset \Delta^{r+1}$ and $\Delta^r \subset \mathcal{D}^r$, and so $\Delta^r \subset \Delta^{r+1} \cap \mathcal{D}^r(R)$.

For the inclusion in the other direction, let $P = \sum_{\beta_1+\cdots+\beta_n \leq r+1} p_{\beta_1\ldots\beta_n}\partial_1^{\beta_1}\cdots\partial_n^{\beta_n} \in \Delta^{r+1}$, and suppose that $P \notin \Delta^r$. Then there exists $\beta^* = (\beta_1^*, \ldots, \beta_n^*)$ such that $\beta_1^* + \cdots + \beta_n^* = r + 1$ and $p_{\beta_1^*\ldots\beta_n^*} \neq 0$. But then $\mathrm{ad}_{\xi_n}^{\beta_n^*} \cdots \mathrm{ad}_{\xi_1}^{\beta_1^*} P = p_{\beta_1^*\ldots\beta_n^*} \neq 0$, and hence $P \notin \mathcal{D}^r(R)$. $\square$

Lemma 3.4.9. *Let* $D \in \mathcal{D}(R)$ *be such that* $[D, \xi_i] = 0$ *for* $i = 1 \ldots n$, *then* $D \in \mathcal{D}^0(R) = R$.

Proof. It is clear from the properties of $[,]$ that in this case, $[D, \sigma] = \delta_\sigma D = 0$ for all $\sigma \in R$, and so by definition $D \in \mathcal{D}^0(R)$. $\square$

Lemma 3.4.10. *Let* $P_1, \ldots, P_n \in \Delta^{r-1}$ *such that*

(i) P_i *is* ξ_i*-integrable for* $i = 1 \ldots n$.

(ii) $[P_i, \xi_j] = [P_j, \xi_i]$ *whenever* $1 \leq i, j \leq n$.

Then there exists $Q \in \Delta^r$ *such that* $P_i = [Q, \xi_i]$, *for* $i = 1, \ldots, n$.

Proof. Assume we have found $Q' \in \Delta^r$ such that $[Q', \xi_i] = P_i$ for $i = k+1, \dots n$, and write $G = [Q', \xi_k] - P_k$. Then G is ξ_k-integrable, as both $[Q', \xi_k]$ and P_k are.

Now for $i = k+1, \dots, n$, $[G, \xi_i] = [[Q', \xi_k] - P_k, \xi_i] = [[Q', \xi_k], \xi_i] - [P_k, \xi_i] = [[Q', \xi_i], \xi_k] - [P_k, \xi_i]$, and hence

$$[G, \xi_i] = 0, \;\; i = k+1, \dots, n$$

Using lemma 3.4.4(v) repeatedly, we see that G can be written as

$$G = \sum_\beta g_\beta \partial^\beta$$

where $g_\beta \in R$, and $\beta_i = 0$ for $i = k+1, \dots, n$.

Now recall that G is ξ_k-integrable, and define

$$Q'' = \sum_\beta \varepsilon(k, \beta)(\beta_k + 1)^{-1} g_\beta \partial^{\beta + e_k}$$

and note that by definition $[Q'', \xi_k] = G$ (if ξ_k is even, this is immediate, while if ξ_k is odd, recall that integrable means $\beta_k = 0$).

However $Q' \in \Delta^r \subset \mathcal{D}^r(R)$ implies by definition that

$$[Q', \xi_k] \in \Delta^r \cap \mathcal{D}^{r-1}(R) = \Delta^r$$

Since $P_k \in \Delta^{r-1}$, then so does $G = [Q', \xi_k] - P_k$, in which case $Q'' \in \Delta^r$. On the other hand $[Q'', \xi_i] = 0$ for $i = k+1, \dots, n$ by construction.

But then $[Q' - Q'', \xi_i] = [Q', \xi_i] = P_i$ for $i = k+1, \dots, n$, while

$$[Q' - Q'', \xi_k] = [Q', \xi_k] - G = P_k$$

Hence $Q' - Q'' \in \Delta^r$, and $[Q' - Q'', \xi_i] = P_i$ for $i = k, \dots, n$, and the result follows by induction, as the base case $k = n$ holds since P_n is assumed to be ξ_n-integrable. $\square$

We can now state and prove:

Theorem 3.4.11. *Let R be a free graded commutative algebra, $\mathcal{D}(R)$ its algebra of differential operators and $\Delta(R)$ the sub-algebra generated by R and its graded derivations.*

Then $\mathcal{D}^k(R) = \Delta^k(R)$ for all $k \geq 0$. In particular $\mathcal{D}(R) = \Delta(R)$ and $\mathcal{D}(R)$ is generated in degree 0 and 1, by R and its graded derivations $\underline{\mathrm{Der}}(R)$.

Proof. It is enough to show that $\mathcal{D}^k(R) \subset \Delta^k(R)$. Note also that we already know that $\mathcal{D}^0 = \Delta^0$ and $\mathcal{D}^1 = \Delta^1$.

Now suppose, by induction, that $\mathcal{D}^k = \Delta^k$ for $k \leq m-1$. Let $P \in \mathcal{D}^m$, and write $P_i := [P, \xi_i]$ for $i = 1, \ldots, n$. Since $P_i \in D^{m-1}$ by definition, it follows by induction that $P_i \in \Delta^{m-1}$. But for all $i, j = 1, \ldots, n$

$$[P_i, \xi_j] = [[P, \xi_i], \xi_j] = [[P, \xi_j], \xi_i] = [P_j, \xi_i]$$

Since each P_i is ξ_i-integrable, Lemma 3.4.10 proves the existence of $Q \in \Delta^m$ such that $[Q, \xi_i] = P_i$. But then $[Q - P, \xi_i] = 0$ for $i = 1, \ldots, n$ and so by lemma 3.4.9, $Q - P \in R = \Delta^0 \in \Delta^m$, and hence $P \in \Delta^m$ and $\mathcal{D}^m \subset \Delta^m$. $\square$

3.5 Differential operators on differential forms

Let Spec(A) be a nonsingular affine variety. Then the algebra ΩA of (commutative) differential forms of A is a graded commutative algebra, and one can indeed consider the ring of differential operators $\mathcal{D}(\Omega A)$ as defined in the previous section.

In light of the previous section, to help identify this ring, one needs to first identify the space of graded derivations of the algebra ΩA. In the context of smooth manifolds, this space was completely described by Frölicher and Njenhuis in [FN56]. They show that any graded derivation of ΩX where X is a C^∞-smooth manifold is the sum of an insertion operator and a Lie derivative. We carry here a similar argument in the algebraic setting, and we identify those derivations.

Let A be as before, ΩA be its algebra of (commutative) differential forms, and $\mathrm{Der}(A)$ be its module of derivations. We define two special types of derivations of ΩA.

Definition 3.5.1. Let K be a *vector valued* k-form on A, i.e. $K \in \mathrm{Der}(A) \otimes \Omega^k A$, then:

(i) The *insertion operator* ι_K is the derivation of ΩA of degree $k-1$ defined as follows. For $\omega \in \Omega^l A$ ($l \geq 1$), $\iota_K \omega \in \Omega^{k+l-1} A$ is given by:

$$\iota_K \omega(X_1, \cdots, X_{k+l-1}) := \frac{1}{k!(l-1)!} \sum_{\sigma \in S_{k+l-1}} (-1)^\sigma$$
$$\omega(K(X_{\sigma(1)}, \cdots, X_{\sigma(k)}), X_{\sigma(k+1)}, \cdots, X_{\sigma(k+l-1)})$$

and for $f \in \Omega^0 A$,

$$\iota_K f = 0$$

(ii) the *Lie derivative* $\mathcal{L}_K$ is the derivation of ΩA of degree k defined by

$$\mathcal{L}_K \;:=\; [d, \iota_K] = d\iota_K - (-1)^{k-1} \iota_K d$$

Note 3.5.2. if $K \in \mathrm{Der}(A) \otimes \Omega^0 A \simeq \mathrm{Der}(A)$, i.e. a regular derivation, the previous definitions reduce to the standard *interior product* i_K and *Lie derivative* L_K.

Note also that any derivation of ΩA is entirely determined by its action on A and $\Omega^1 A$ by the Leibniz rule and the universal property of ΩA. This shows that there are no non-zero derivations of ΩA of degree $r \leq -2$.

Example 3.5.3. Consider now A to be the affine line with coordinate x, and let $K = \partial_x dx \in \mathrm{Der}(A) \otimes \Omega^1 A$. Then ι_K is a derivation of ΩA of degree 0, and for $\omega = \alpha dx \in \Omega^1 A$ and $\xi = \beta \partial_x$, we have

$$\begin{aligned} \iota_K \omega(\xi) &= \omega(K(\xi)) \\ &= \omega(\beta \partial_x) \\ &= \alpha\beta \\ &= \omega(\xi) \end{aligned}$$

In other words, $\iota_K(\omega) = \omega$ for $\omega \in \Omega^1 X$, but yet $\iota_K(f) = 0$ for $f \in \mathbb{C}[x]$ (this is nothing but the Euler derivation on ΩA, really).

On the other hand, $\mathcal{L}_K = d\iota_K - \iota_K d = -d$.

Note that in this case $\iota_K = dx \wedge i_{\frac{\partial}{\partial x}}(-)$, and one can express the insertion operator in terms of the standard interior product.

We now focus on the special case where $A = \mathbb{C}[V]$ and V is a finite dimensional vector space. In this case, $\Omega A \simeq \mathrm{Sym}(V^*) \otimes \bigwedge V^*$ is a free graded commutative algebra and thus we already know the structure of $\underline{\mathrm{Der}}\, \Omega A$ from Lemma 3.4.2. We now discuss how that relates to the insertion operators and Lie derivative derivations as defined by Frölicher and Njenhuis.

Proposition 3.5.4. *Let V be a finite dimensional vector space and let $A = \mathbb{C}[V]$. Then* *(i) Every graded derivation $D \in \underline{\mathrm{Der}}(\Omega A)$ of degree $r \geq -1$ that vanishes on A is an insertion operator.*

(ii) Every graded derivation $D \in \underline{\mathrm{Der}}(\Omega A)$ of degree $r \geq -1$ that commutes with d is a Lie derivative (including d itself).

(iii) Every graded derivation of ΩA has a unique decomposition into a sum of an insertion operator and a Lie derivative.

Proof. (i) Pick orthonormal dual bases $(\xi_i)_{i=1..n}$ and $(dx^i)_{i=1..n}$ for $\mathrm{Der}(A)$ and $\Omega^1(A)$, i.e. $dx^i(\xi_j) = \delta^i_j$, and consider the vector valued $(1+r)$-form

$$K := \sum_j \xi_j \otimes D(dx^j) \in \mathrm{Der}(A) \otimes \Omega^{1+r}(A)$$

Then we have the following for $i = 1 \ldots n$ and $\eta_1, \ldots, \eta_{1+r} \in \mathrm{Der}(A)$:

$$\begin{aligned}
\iota_K dx^i(\eta_1, \ldots, \eta_{1+r}) &= dx^i(K(\eta_1, \ldots, \eta_{1+r})) \\
&= dx^i(\sum_j \xi_j D(dx^j)(\eta_1, \ldots, \eta_{1+r})) \\
&= \sum_j D(dx^j)(\eta_1, \ldots, \eta_{1+r}) dx^i(\xi_j) \\
&= D(dx^i)(\eta_1, \ldots, \eta_{1+r})
\end{aligned}$$

In other words, we have:

$$\iota_K dx^i = D(dx^i)$$

And so for a homogeneous (wlog) $w = w_{i_1 \ldots i_m} dx^{i_1} \wedge \cdots \wedge dx^{i_m} \in \Omega^m(A)$, we have $D(w) = w_{i_1 \ldots i_m} D(dx^{i_1} \wedge \cdots \wedge dx^{i_m})$ because D vanishes on A. But using the Leibniz rule and the identity shown above, it is now clear that

$$D = \iota_K$$

(ii) For $\eta_1, \ldots, \eta_{1+r} \in \mathrm{Der}(A)$, it is easy to see that $D|_A(-)(\eta_1, \ldots, \eta_{1+r})$ is a derivation of A, and so the mapping $(\eta_1, \ldots, \eta_{1+r}) \mapsto D|_A(-)(\eta_1, \ldots, \eta_{1+r})$

defines a vector valued $(r+1)$-form K, and by definition for $f \in A$,

$$\begin{aligned} D(f)(\eta_1, \dots, \eta_{1+r}) &= (K(\eta_1, \dots, \eta_{1+r}))(f) \\ &= i_{K(\eta_1, \dots, \eta_{1+r})} df \\ &= \iota_K(df)(\eta_1, \dots, \eta_{1+r}) \\ &= \mathcal{L}_{(-1)^r K} f \end{aligned}$$

One ends the proof by realizing that D and $\mathcal{L}_{(-1)^r K}$ are two derivations of ΩA that commute with d and agree on A, and so they have to be equal.

Notice that if $K = \sum_i dx^i \otimes \xi_i$, then $\mathcal{L}_K = -d$.

(iii) Consider the Lie derivative $\mathcal{L}$ generated by the restriction of D to A, then $D - \mathcal{L}$ is easily checked to be an insertion operator ι and $D = \mathcal{L} + \iota$.

□

We now have yet another proof for the following statement:

Proposition 3.5.5. *Let $A = \mathbb{C}[V]$, where V is a finite dimensional vector space. Then the module $\underline{\mathrm{Der}}(\Omega A)$ is generated over ΩA by*

(i) the insertions operators $\iota_K = i_K$ for $K \in \mathrm{Der}(A) \otimes \Omega^0 A \simeq \mathrm{Der}(A)$

(ii) the Lie derivatives $\mathcal{L}_K = L_K$ for $K \in \mathrm{Der}(A) \otimes \Omega^0 A \simeq \mathrm{Der}(A)$

i.e. regular interior products and Lie derivatives of forms.

Remark 3.5.6. Notice that we do **not** need to include d.

Proof. We know that $\underline{\mathrm{Der}}(\Omega A)$ is generated by the insertion operators and Lie derivatives of all degree. We show that in this case, those operators can by written in terms of the standard inner and Lie operators.

Let $K = \sum_i \xi_i \otimes \omega^i \in \mathrm{Der}(A) \otimes \Omega^k A$, then we claim that $\iota_K = \sum_i \omega^i \wedge (i_{\xi_i}(-))$

The general proof follows this example. Say $K = \xi \otimes \omega$ for $\omega \in \Omega^1(A)$ and $\Omega \in \Omega^2(A)$, then

$$\begin{aligned}
(\iota_K \Omega)(X,Y) &= \Omega(K(X),Y) + \Omega(X,K(Y)) \\
&= \Omega(\xi\omega(X),Y) + \Omega(X,\xi\omega(Y)) \\
&= w(X)\Omega(\xi,Y) + \omega(Y)\Omega(X,\xi)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
(\omega \wedge i_\xi \Omega)(X,Y) &= i_Y i_X(\omega \wedge i_\xi \Omega) \\
&= i_Y((i_X\omega) \wedge i_\xi\Omega) + i_Y((-1)^1\omega \wedge i_X i_\xi \Omega) \\
&= (i_Y i_X \omega) \wedge i_\xi \Omega + (-1)^0 i_X\omega \wedge i_Y i_\xi \Omega \\
&\quad -(i_Y\omega) \wedge i_X i_\xi \Omega - (-1)^1 \omega \wedge i_Y i_X i_\xi \Omega \\
&= i_X\omega \wedge i_Y i_\xi \Omega - (i_Y\omega) \wedge i_X i_\xi \Omega \\
&= \omega(X)\Omega(\xi,Y) - \omega(Y)\Omega(\xi,X) \\
&= \omega(X)\Omega(\xi,Y) + \omega(Y)\Omega(X,\xi)
\end{aligned}$$

Also

$$
\begin{aligned}
\mathcal{L}_K &= [d, \iota_K] \\
&= d(w \wedge i_\xi) - (-1)^{k-1} w \wedge (i_K d) \\
&= (dw) \wedge i_\xi + (-1)^k w \wedge d i_\xi - (-1)^{k-1} w \wedge (i_K d) \\
&= (dw) \wedge i_\xi + (-1)^k w \wedge (d i_{xi} + i_{xi} d) \\
&= (dw) \wedge i_\xi + (-1)^k w \wedge L_\xi
\end{aligned}
$$

□

Note 3.5.7. Each element $\xi \in V$ defines an element in $\mathrm{Der}(A)$, written ξ as well, by extending the duality bracket $\langle -, \xi \rangle$. In fact such derivations generate $\mathrm{Der}(A)$ over A. Then $i_\xi \in \underline{\mathrm{Der}}(\Omega V)$ is given explicitely for $\xi \in V$ and $x \in V^*$, then

$$i_\xi dx = \langle x, \xi \rangle$$

Note also that as the commutator of two derivations of A, $[\xi, x] = \langle x, \xi \rangle$.

Finally, we have the following description of $\mathcal{D}(\Omega A)$ for $A = \mathbb{C}[V]$.

Theorem 3.5.8. *Let $A = \mathbb{C}[V]$, where V is a finite dimensional vector space. Then $\mathcal{D}(\Omega A)$ is generated as an algebra over $\mathbb{C}$ by $\underline{\mathrm{Der}}(\Omega A)$ and ΩA.*

More precisely, let $\{\xi_i\}_{i=1\ldots n}$ be a basis of V, and $\{x_i\}_{i=1\ldots n}$ the corresponding dual basis of V^*. Then $\mathcal{D}(\Omega A)$ is generated by

(i) Lie derivatives L_{ξ_i} and multiplication operators m_{xi}, for $i = 1 \ldots n$

(ii) insertion operators i_{ξ_i} and multiplication operators m_{dx_i}, for $i = 1 \ldots n$

Proof. This follows from the fact that $\Omega A = \operatorname{Sym} V^* \otimes \bigwedge V^*$ is a free graded commutative algebra, and thus we can use Theorem 3.4.11 to conclude that $\mathcal{D}(\Omega A)$ is generated in degree 1 by multiplication operators and graded derivations. Finally, Proposition 3.5.5 describes all such derivations. □

In fact, we can do better than the previous theorem suggests. Recall the definition of the Clifford algebra on $V \oplus V^*$. The vector space $V \oplus V^*$ has a canonical symmetric bilinear form $\langle , \rangle_{Cl}$ on $V \oplus V^*$, given by

$$\langle \xi + x, \eta + y \rangle_{Cl} := \frac{1}{2} \left(\langle x, \eta \rangle + \langle y, \xi \rangle \right)$$

where again, $\langle , \rangle$ is the duality $V^* \times V \mapsto \mathbb{C}$

The Clifford algebra $Cl(V \oplus V^*)$ of $V \oplus V^*$ with the canonical bilinear form is then quotient of the tensor algebra $T(V \oplus V^*)$ by the ideal generated by the relations of the form ($x, y \in V^*$, $\xi, \eta \in V$):

$$\begin{aligned} \xi\eta &= -\eta\xi \qquad (\text{and so } \xi^2 = 0) \\ xy &= -yx \qquad (\text{and so } x^2 = 0) \\ \xi x + x\xi &= \langle x, \xi \rangle \end{aligned}$$

Theorem 3.5.9. *There is an isomorphism of algebras*

$$\mathcal{D}(\Omega V) \simeq \mathcal{D}(V) \otimes Cl(V \oplus V^*)$$

Proof. It is very easy to check that, using the notations of the previous theorem, L_{ξ_i} and m_{x_i} generate a subalgebra of $\mathcal{D}(\Omega V)$ isomorphic to $\mathcal{D}(V)$, while i_{ξ_i} and m_{dx_i} generate a subalgebra isomorphic to $Cl(V \oplus V^*)$, and finally that those two commute.

Explicitly, recall that $\mathcal{D}(V)$ is generated by multiplication operators m_x for $x \in V^*$, and derivations in $\mathrm{Der}(\mathbb{C}[V])$. Recall also that $\mathrm{Der}(\mathbb{C}[V])$ is generated by the derivations ∂_ξ for $\xi \in V$ which extend $\langle -, \xi \rangle$ to $\mathbb{C}[V^*]$. Then the map defined on generators by

$$\mathcal{D}(V) \otimes Cl(V \oplus V^*) \longrightarrow \mathcal{D}(\Omega V)$$

$$m_x \otimes 1 \mapsto m_x$$

$$\partial_\xi \otimes 1 \mapsto L_\xi$$

$$1 \otimes x \mapsto m_{dx}$$

$$1 \otimes \xi \mapsto i_\xi$$

is well defined and provides the required isomorphism. $\square$

3.6 Equivariant differential operators on forms

Let now W be a finite Coxeter group acting on its finite dimensional reflection representation V. Let also V^* be the dual of V with the natural action of W, and write $\langle , \rangle$ for the canonical pairing $V^* \times V \to \mathbb{C}$.

Note 3.6.1. We will always reserve Roman letters for the elements in V^*, and Greek letters for the elements in V.

Finally, let now $\Sigma \subset W$ to be the set of all (real) reflections in W. For each $s \in \Sigma$, let $\alpha_s \in V^*$ define the reflection hyperplane of s, and $\alpha_s^\vee \in V$ be the (-1)-eigenvector such that $\langle \alpha_s, \alpha_s^\vee \rangle = 2$. Recall that for $x \in V^*$ and $\xi \in V$:

$$\begin{aligned} (1-s)x &= \langle x, \alpha_s^\vee \rangle \alpha_s \\ (1-s)\xi &= \langle \alpha_s, \xi \rangle \alpha_s^\vee \end{aligned}$$

The action of W on V naturally defines an action of W on V^* which extend to $\mathbb{C}[V] := \mathrm{Sym}(V^*)$, $\mathbb{C}[V^*] := \mathrm{Sym}(V)$ and $\bigwedge V^*$.

Now consider $\Omega V := \mathrm{Sym}(V^*) \otimes \bigwedge V^*$, the algebra of (commutative) differential forms on the affine space V, and $\underline{\mathrm{End}}(\Omega V)$ the algebra of graded ($\mathbb{C}$-linear) endomorphisms of ΩV. We can extend the action of W to ΩV in three different ways: first, we can take the tensor product of the trivial representation on $\mathrm{Sym}(V^*)$ and the natural representation on $\bigwedge V^*$; second, we can take the tensor product of the natural representation on $\mathrm{Sym}(V^*)$ and the trivial one on $\bigwedge V^*$; and finally we can take the diagonal action of W on ΩV. More precisely:

Definition 3.6.2. For $s \in W$, we write $s \otimes 1$, $1 \otimes s$ and $s = s \otimes s$ respectively, for the 3 different endomorphisms of ΩV given for $f \in \mathbb{C}[V]$ and $g_i \in V^*$, by the following formulas:

$$s(f dg_1 \wedge \cdots \wedge dg_n) := {}^s f d({}^s g_1) \wedge \cdots \wedge d({}^s g_n)$$
$$(s \otimes 1)(f dg_1 \wedge \cdots \wedge dg_n) := {}^s f dg_1 \wedge \cdots \wedge dg_n$$
$$(1 \otimes s)(f dg_1 \wedge \cdots \wedge dg_n) := f d({}^s g_1) \wedge \cdots \wedge d({}^s g_n)$$

We let $\mathbb{C}W$ act on ΩV by extending the above actions by linearity, and we get three different representations $\mathbb{C}W \to \underline{\mathrm{End}}(\Omega V)$.

Note that by definition, we have that $s(dg) = d({}^s g)$, in other words, acting diagonally on differential forms we have:

$$sd = ds$$

Consider now the algebra $\mathcal{D}(\Omega V)$ of differential operators on the graded commutative algebra ΩV. The diagonal action of W on ΩV yields a natural action of

W on $\mathcal{D}(\Omega V)$, such that for $s \in W$ and $D \in \mathcal{D}(\Omega V)$,we have

$$s \cdot D := sDs^{-1}$$

where the left and right $\mathbb{C}W$-module structures come from $\mathcal{D}(\Omega V) \subset \underline{\mathrm{End}}(\Omega V)$. Explicitely,

$$(s \cdot D)(\omega) = s \cdot D(s^{-1} \cdot \omega)$$

for $s \in W$ and $\omega \in \Omega V$. This allows us to form the crossed product algebra $\mathcal{D}(\Omega V) \rtimes W$ of *equivariant differential operators on differential forms* on V. This algebra turns out to be very closely related to the algebra of equivariant differential operators on V. We explain the nature of this relationship in the rest of this section.

First, we investigate the relationship between the three different actions of W on ΩV.

Proposition 3.6.3. *The mapping $\mathbb{C}W \to Cl(V \oplus V^*)$ given on generating reflections by*

$$s \in W \mapsto \Delta_s := \frac{1}{2}\left(\alpha_s^\vee \alpha_s - \alpha_s \alpha_s^\vee\right) \in Cl(V \oplus V^*) \tag{3.6}$$

is a well defined homomorphism of algebras, and the natural action of W on $Cl(V \oplus V^*)$ becomes conjugation inside $Cl(V \oplus V^*)$. In other words,

$$\Delta_s x \Delta_s = {}^s x$$

$$\Delta_s \xi \Delta_s = {}^s \xi$$

holds inside $Cl(V \oplus V^*)$ for $x \in V^*$, $\xi \in V$ and $s \in \Sigma$.

Proof. First, we check that Δ_s squares to 1 in $Cl(V \oplus V^*)$. Indeed

$$\begin{aligned}
\Delta_s^2 &= \frac{1}{4}(\alpha_s^\vee \alpha_s - \alpha_s \alpha_s^\vee)^2 \\
&= \frac{1}{4}(\alpha_s^\vee \alpha_s \alpha_s^\vee \alpha_s + \alpha_s \alpha_s^\vee \alpha_s \alpha_s^\vee) \\
&= \frac{1}{4}(\alpha_s^\vee (2 - \alpha_s^\vee \alpha_s)\alpha_s + \alpha_s(2 - \alpha_s \alpha_s^\vee)\alpha_s^\vee) \\
&= \frac{1}{2}(\alpha_s^\vee \alpha_s + \alpha_s \alpha_s^\vee) = 1
\end{aligned}$$

Next, an easy but lengthy computation shows that for two reflections s and t, sts is also a reflection and

$$\Delta_s \Delta_t \Delta_s = \Delta_{sts} \tag{3.7}$$

Now recall that W being a Coxeter group, it has a presentation of the form $\langle s_1, \dots s_n | (s_i s_j)^{n_{ij}} = 1 \rangle$, where $n_{ii} = 1$ and $n_{ij} \geq 2$ for $i \neq j$. To show that the map defined on generators by (3.6) is well defined, we just need to show that these relations still hold when we replace s_i and s_j by Δ_{s_i} and Δ_{s_j}. We do the computation for two reflections s and t such that $(st)^2 = 1$, the general case following from a similar computation. Note that (3.7) implies that $\Delta_t \Delta_s = \Delta_s \Delta_{sts}$, since $\Delta_s^2 = 1$. Then

$$\begin{aligned}
(\Delta_s \Delta_t)^2 &= \Delta_s \Delta_t \Delta_s \Delta_t \\
&= \Delta_s \Delta_s \Delta_{sts} \Delta_t \\
&= \Delta_{sts} \Delta_t \\
&= \Delta_t \Delta_{tstst} = \Delta_t \Delta_t = 1
\end{aligned}$$

Finally, note that $\alpha_s^\vee \alpha_s - 1 = \frac{1}{2}(2\alpha_s^\vee\alpha_s - 2) = \frac{1}{2}(\alpha_s^\vee\alpha_s - \alpha_s\alpha_s^\vee)$

$$\begin{aligned}
\Delta_s \xi \Delta_s &= (\alpha_s^\vee\alpha_s - 1)\xi(\alpha_s^\vee\alpha_s - 1) \\
&= (\alpha_s^\vee\alpha_s\xi - \xi)(\alpha_s^\vee\alpha_s - 1) \\
&= (\alpha_s^\vee(\langle \alpha_s, \xi\rangle - \xi\alpha_s) - \xi)\,(\alpha_s^\vee\alpha_s - 1) \\
&= (\langle \alpha_s, \xi\rangle\alpha_s^\vee + \xi\alpha_s^\vee\alpha_s - \xi)(\alpha_s^\vee\alpha_s - 1) \\
&= \langle \alpha_s, \xi\rangle\alpha_s^\vee\alpha_s^\vee\alpha_s + \xi\alpha_s^\vee\alpha_s\alpha_s^\vee\alpha_s - \xi\alpha_s^\vee\alpha_s - \langle \alpha_s, \xi\rangle\alpha_s^\vee - \xi\alpha_s^\vee\alpha_s + \xi \\
&= \xi\alpha_s^\vee\alpha_s\alpha_s^\vee\alpha_s - 2\xi\alpha_s^\vee\alpha_s - \langle \alpha_s, \xi\rangle\alpha_s^\vee + \xi \\
&= \xi\alpha_s^\vee(2 - \alpha_s^\vee\alpha_s)\alpha_s - 2\xi\alpha_s^\vee\alpha_s - \langle \alpha_s, \xi\rangle\alpha_s^\vee + \xi \\
&= \xi - \langle \alpha_s, \xi\rangle\alpha_s^\vee \\
&= {}^s\xi
\end{aligned}$$

and similarly for $x \in V^*$. $\square$

Proposition 3.6.4. *For $s \in \Sigma$, the operators s, $s \otimes 1$ and $1 \otimes s$ satisfy:*

$$s \otimes 1 = s(i_{\alpha_s^\vee} m_{d\alpha_s} - 1) = \frac{1}{2} s\left(i_{\alpha_s^\vee} m_{d\alpha_s} - m_{d\alpha_s} i_{\alpha_s^\vee}\right) \tag{3.8}$$

$$1 \otimes s = (i_{\alpha_s^\vee} m_{d\alpha_s} - 1) = \frac{1}{2}\left(i_{\alpha_s^\vee} m_{d\alpha_s} - m_{d\alpha_s} i_{\alpha_s^\vee}\right) \tag{3.9}$$

In particular, the operators s, $s \otimes 1$ and $1 \otimes s$ all belong to $\mathcal{D}(\Omega V) \rtimes W$.

Proof. This follows from Theorem 3.5.9 and Proposition 3.6.3, as the the operators i_ξ and m_{dx} generate an algebra isomorphic to $Cl(V \oplus V^*)$. As a reality check, we provide the following computation. Let $\omega = f dz_1 \wedge \cdots \wedge dz_k$, where $f \in \mathbb{C}[V]$ and

z_i's in V^*. Then

$$\begin{aligned}
(i_{\alpha_s^\vee} m_{d\alpha_s} - 1)\cdot\omega &= i_{\alpha_s^\vee}(f d\alpha_s dz_1 \wedge \cdots \wedge dz_k) - w \\
&= (\langle \alpha_s, \alpha_s^\vee \rangle - 1)\omega - \sum_i (-1)^i f d\alpha_s dz_1 \cdots (i_{\alpha_s^\vee} dz_i) \cdots dz_k \\
&= \omega - \sum_i f dz_1 \cdots (\langle z_i, \alpha_s^\vee \rangle d\alpha_s) \cdots dz_k \\
&= \sum_i f d(z_1 - \langle z_1 \alpha_s^\vee \rangle \alpha_s) \cdots d(z_k - \langle z_k \alpha_s^\vee \rangle \alpha_s) \\
&= (1 \otimes s)\cdot\omega
\end{aligned}$$

The other identities follow from the fact that $s = (s \otimes 1)(1 \otimes s)$, and that they all square to 1. □

We are ready to prove the two main theorems of this section that help to better understand the structure of the algebra of equivariant differential operators on forms.

Theorem 3.6.5. *Let W be a finite Coxeter group acting on its finite dimensional reflection representation V, then*

$$\mathcal{D}(\Omega V) \rtimes W \simeq (\mathcal{D}(V) \rtimes W) \otimes Cl(V \oplus V^*) \tag{3.10}$$

Proof. The map

$$\begin{aligned}
(\mathcal{D}(V) \rtimes W) \otimes Cl(V \oplus V^*) &\longrightarrow \mathcal{D}(\Omega V) \rtimes W \\
s \otimes 1 &\mapsto \frac{1}{2} s \left(i_{\alpha_s^\vee} m_{d\alpha_s} - m_{d\alpha_s} i_{\alpha_s^\vee} \right) \\
m_x \otimes 1 &\mapsto m_x \\
\partial_\xi \otimes 1 &\mapsto L_\xi \\
1 \otimes x &\mapsto m_{dx} \\
1 \otimes \xi &\mapsto i_\xi
\end{aligned}$$

provides the required isomorphism. □

Theorem 3.6.6. *Let V and W as in the theorem above. Then $\mathcal{D}(\Omega V) \rtimes W$ is Morita equivalent to $\mathcal{D}(V) \rtimes W$, and so their Hochschild cohomology groups agree, in other words*

$$HH^\bullet(D(\Omega V) \rtimes W) \cong HH^\bullet(\mathcal{D}(V) \rtimes W) \tag{3.11}$$

Proof. We showed that $Cl(V \oplus V^*)$ is a matrix algebra, and hence $\mathcal{D}(\Omega V) \rtimes W \simeq (\mathcal{D}(V) \rtimes W) \otimes Cl(V \oplus V^*)$ is isomorphic to a matrix algebra over $\mathcal{D}(V) \rtimes W$. The claim about the Morita equivalence and Hochschild cohomology follow.

□

In particular, by Theorem 2.4.3, we have the following corollary.

Corollary 3.6.7. *$HH^2(\mathcal{D}(\Omega V) \rtimes W) = \mathbb{C}[\Sigma]^W$, where $\Sigma \subset W$ is the subset of reflections, $HH^1(\mathcal{D}(\Omega V) \rtimes W) = HH^3(\mathcal{D}(\Omega V) \rtimes W) = 0$, and there exists a universal deformation of $\mathcal{D}(\Omega V) \rtimes W$ parametrized by $\mathbb{C}[\Sigma]^W$, the conjugation invariant functions on the set of reflections.*

The question of whether one can realize a universal deformation algebraically is a natural one and we will answer it affirmatively in the next chapter. Before we move on to describing such a deformation, we mention here an easy result that describes formal deformations of matrix algebras.

Let A_ϵ be a formal deformation of A with product $\star : A[[\epsilon]] \otimes_{\Bbbk[[\epsilon]]} A[[\epsilon]] \to A[[\epsilon]]$. The underlying vector space of A_ϵ is $A[[\epsilon]]$ and it is easy to check that $M_n(A[[\epsilon]]) \simeq M_n(A)[[\epsilon]]$ as vector spaces. Then $M_n(A[[\epsilon]])$ is a deformation of $M_n(A) \simeq A \otimes M_n(\Bbbk)$, where the product is defined for $a, b \in A$ and $M, N \in M_n(\Bbbk)$ by $aM \star bN := (a \star b)MN$. Conversely:

Proposition 3.6.8 ([MPU09], Proposition C.1)**.** *Let A be an algebra and $M_n(A)$ be the algebra of $n \times n$ matrices over A. Then any formal deformation of $M_n(A)$ is equivalent to a deformation $M_n(A_\epsilon)$, where A_ϵ is a formal deformation of A.*

CHAPTER 4

THE GRADED CHEREDNIK ALGEBRA

Before we move on to the definition of the graded Cherednik algebra, we reinterpret the previous results on differential operators on forms using the language of graded algebras. In this chapter, we assume $\Bbbk = \mathbb{C}$ for simplicity.

4.1 Notation

Recall that we can view any vector space V as a graded vector space with a single component in degree 0. In this case $V[1]$ is simply the graded vector space with V sitting in degree -1, and 0 everywhere else, while $(V[1])^* = (V^*)[-1]$, i.e. V^* sitting in degree 1. The graded vector space $V^{\otimes n}$ is simply $V^{\otimes n}$ sitting in degree 0, while $(V[1])^{\otimes n}$ is $(V^{\otimes n})[n]$, in other words, $V^{\otimes n}$ sitting in degree $-n$. From this, it follows that the graded tensor algebra TV is simply the algebra TV sitting in degree 0, while $T(V[1])$ is the algebra TV where $V \subset TV$ is given degree -1.

Following the definition, it is easy to check that $\underline{Sym}(V^*) = \mathrm{Sym}(V^*)$, the standard symmetric algebra on V^*, sitting in degree 0, in others words $\underline{Sym}(V^*) = \mathbb{C}[V]$, the algebra of regular functions on V. On the other hand, $\underline{Sym}\,[(V[1])^*] = \bigwedge V^*$ is the exterior algebra on V^*, where the grading comes from giving V^* degree 1.

Consider now the graded vector space $\mathbb{V} := V \oplus V[1]$. Its dual $\mathbb{V}^* = (V \oplus V[1])^* = V^* + V^*[-1]$ has two copies of V^* in degree 0 and 1. Then an easy computation shows that

$$\underline{Sym}\,\mathbb{V}^* = \mathrm{Sym}\,V^* \otimes \bigwedge V^*$$

where $V^* \otimes 1$ has degree 0, and $1 \otimes V^*$ has degree 1. In other words

$$\underline{Sym}\,\mathbb{V}^* = \Omega V$$

with the standard grading on ΩV.

Remark 4.1.1. This agrees with a well known "supergeometric" interpretation of the algebra ΩM of differential forms on a smooth manifold as the algebra of functions on the shifted tangent bundle $T[1]M$.

Let now $V^\bullet$ be a graded vector space.

Definition 4.1.2. The algebra of differential operators $\mathcal{D}(V^\bullet)$ on $V^\bullet$ is defined to be

$$\mathcal{D}(V^\bullet) := \mathcal{D}\left(\underline{Sym}\,[(V^\bullet)^*]\right)$$

where $\mathcal{D}\left(\underline{Sym}\,[(V^\bullet)^*]\right)$ is the algebra of differential operators on the graded commutative algebra $\underline{Sym}\,[(V^\bullet)^*]$ as we defined in Chapter 2.

It follows immediately that when V is a vector space sitting in degree 0, this definition agrees with the standard one. Also we now have

$$\mathcal{D}(\mathbb{V}) = \mathcal{D}(\underline{Sym}\,\mathbb{V}^*) = \mathcal{D}(\Omega V)$$

4.2 The graded Cherednik algebra

Let now W be a finite reflection group. Recall that the rational Cherednik algebra H_k represents the universal deformation of $\mathcal{D}(V) \rtimes W$. For a finite Coxeter group

W, recall that H_k was defined as the quotient of the algebra $T(V \oplus V^*) \rtimes W$, which is the crossed product of the tensor algebra $T(V \oplus V^*)$ with W, by the relations

$$[x, x'] = 0,\ [\xi, \xi'] = 0,\ wxw^{-1} = w(x),\ w\xi w^{-1} = w(\xi)$$
$$[\xi, x] = \langle x, \xi\rangle - \sum_{s\in\Sigma} k_s \langle x, \alpha_s^\vee\rangle\langle \alpha_s, \xi\rangle s$$

for $x, x' \in V^*$, $\xi, \xi' \in V$ and $w \in W$.

In looking for deformations of $\mathcal{D}(\Omega V) \rtimes W = \mathcal{D}(\mathbb{V}) \rtimes W$, this suggests that we look for them in the form of $H_k(\mathbb{V})$ where $H_k(\mathbb{V})$ is a quotient of $T(\mathbb{V} \oplus \mathbb{V}^*) \rtimes W$.

Let us reinterpret the relations defining the standard Cherednik algebras. First observe that

$$\begin{aligned} \mathbb{V} \oplus \mathbb{V}^* &= (V \oplus V[1]) \oplus (V \oplus V[1])^* \qquad (4.1)\\ &\simeq (V \oplus V[1]) \oplus (V^* \oplus V^*[-1]) \\ &\simeq (V \oplus V^*) \oplus (V[1] \oplus V^*[-1]) \end{aligned}$$

This suggests the following notation. For $\xi \in V$ we will still write ξ for its image in degree 0 inside $\mathbb{V}$, while $\iota\xi$ will denote the element $\xi \in V$ sitting in degree -1 inside $\mathbb{V}$. a similar notation will be used for $x \in V^*$ in degree 0, and $\iota x \in V^*[-1] \subset \mathbb{V}^*$ in degree 1.

Recall also the definition of roots $\alpha_s \in V^*$ and coroots $\alpha_s^\vee \in V$. We will think of them as elements in $\mathbb{V}$ and $\mathbb{V}^*$ sitting in degree 0.We can now give the following definition:

Definition 4.2.1. The *graded Cherednik algebra* $\mathbb{H}_k(W)$ is defined to be the quotient of

$$T(\mathbb{V} \oplus \mathbb{V}^*) \rtimes W$$

by the following relations:

$$
\begin{aligned}
&[\mathbb{V}, \mathbb{V}] = [\mathbb{V}^*, \mathbb{V}^*] = 0 \\
&[\boldsymbol{\xi}, \boldsymbol{x}] = \langle \boldsymbol{x}, \boldsymbol{\xi} \rangle - \sum_{s \in \Sigma} k_s \langle \boldsymbol{x}, \alpha_s^\vee \rangle \langle \alpha_s, \boldsymbol{\xi} \rangle s
\end{aligned}
\tag{4.2}
$$

where $\boldsymbol{\xi} \in \mathbb{V}$, $\boldsymbol{x} \in \mathbb{V}^*$, and W acts trivially on $V[1]$ (and thus also on $V^*[-1]$).

Note that W needs to act trivially on $V[1] \subset \mathbb{V}$ to be a reflection representation of W. This agrees with the reflection formula on V and V^*. Indeed, for $\xi \in V$ for example, the identity

$$
{}^s\xi = \xi - \langle \alpha_s, \xi \rangle \alpha_s^\vee
$$

becomes when applied to $\iota\xi$:

$$
{}^s(\iota\xi) = \iota\xi - \langle \alpha_s, \iota\xi \rangle \alpha_s^\vee = \iota\xi
$$

because $\langle V^*, V[1] \rangle = 0$.

We then have the following important structure theorem:

Theorem 4.2.2 (Structure theorem)**.** *The graded Cherednik algebra $\mathbb{H}_k(W)$ is isomorphic as a graded algebra to the algebra*

$$
H_k(W) \otimes Cl(V \oplus V^*)
$$

where $H_k(W)$ is the standard rational Cherednik algebra of W sitting in degree 0, and $Cl(V \oplus V^*)$ is the Clifford algebra on $V \oplus V^*$ with the canonical symmetric pairing and $\deg(1 \otimes V) = -1$, $\deg(1 \otimes V^*) = 1$.

Remark 4.2.3. One might be surprised at first to see that $\mathbb{H}_k$ is graded. Indeed, the standard grading on $T(V \oplus V^*)$ is *not* preserved when passing to the quotient yielding the Clifford algebra, which is then only filtered (it is in fact also a super-algebra). In this case though, the grading on V and $V^* \subset T(V \oplus V^*)$ makes the Clifford relations homogeneous, and $Cl(V \oplus V^*)$ inherits that grading as well. The proof should help understand the reason for this seemingly odd grading.

Proof. The commutation relations $[\mathbb{V}, \mathbb{V}] = 0$ and $[\mathbb{V}^*, \mathbb{V}^*] = 0$ now read for $x, x' \in V^*$ and $\xi, \xi' \in V$:

$$[x, x'] = 0,\ [\xi, \xi'] = 0,\ [x, \iota x'] = 0,\ [\xi, \iota\xi'] = 0,$$

while $[\boldsymbol{\xi}, \boldsymbol{x}] = \langle \boldsymbol{x}, \boldsymbol{\xi} \rangle - \sum_{s\in\Sigma} k_s \langle \boldsymbol{x}, \alpha_s^\vee \rangle \langle \alpha_s, \boldsymbol{\xi} \rangle s$ yields here for $x \in V^*$ and $\xi \in V$:

$$\begin{aligned}
[\xi, x] &= \xi x - x\xi = \langle x, \xi \rangle - \sum_{s\in\Sigma} k_s \langle x, \alpha_s^\vee \rangle \langle \alpha_s, \xi \rangle s \\
[\xi, \iota x] &= \xi \iota x - \iota x \xi = 0 \\
[\iota\xi, x] &= \iota\xi x - x\iota\xi = 0 \\
[\iota\xi, \iota x] &= \iota\xi\iota x + \iota x \iota\xi = \langle \xi, x \rangle
\end{aligned}$$

and thus $H_k(\mathbb{V})$ is the quotient of

$$(T(V \oplus V^*) \rtimes W) \otimes T(V \oplus V^*)$$

by the relations:

$$\begin{aligned}
&[x \otimes 1, x' \otimes 1] = 0,\ [\xi \otimes 1, \xi \otimes 1'] = 0 \\
&w(x \otimes 1)w^{-1} = w(x) \otimes 1,\ w(xi \otimes 1)w^{-1} = w(\xi) \otimes 1 \\
&[\xi \otimes 1, x \otimes 1] = \langle x, \xi \rangle - \sum_{s\in\Sigma} k_s \langle x, \alpha_s^\vee \rangle \langle \alpha_s, \xi \rangle s \\
&(1 \otimes x)(1 \otimes x') = -(1 \otimes x')(1 \otimes x) \\
&(1 \otimes \xi)(1 \otimes \xi') = -(1 \otimes \xi')(1 \otimes \xi) \\
&[1 \otimes \xi, 1 \otimes x]_+ = \langle x, \xi \rangle
\end{aligned}$$

where the commutators are graded and the degrees are as follows:

$$\deg(V \otimes 1) = 0, \ \deg(V^* \otimes 1) = 0, \ \ \deg(1 \otimes V) = -1, \deg(1 \otimes V^*) = 1$$

which concludes the proof. $\square$

Similar to the non-graded case, we have the following elementary theorem:

Theorem 4.2.4. *The graded Cherednik algebra $\mathbb{H}_k$ satisfies:*

(i) $\mathbb{H}_0 = \mathcal{D}(\Omega V) \rtimes W$.

(ii) PBW property: the linear map $\Omega V \otimes \mathbb{C}W \otimes \Omega V^* \to \mathbb{H}_k$ induced by multiplication in $\mathbb{H}_k$ is a left ΩV-module isomorphism.

(iii) The family $\{\mathbb{H}_k\}$ is the universal deformation of $\mathbb{H}_0$.

In fact for both the *standard filtration* on $\mathbb{H}_k$ (where $\deg V = \deg V^* = 1$ on either side and $\deg \mathbb{C}W = 0$) and the *differential filtration* (where $\deg V^* = 0$, $\deg V = 1$ on either side, and $\deg \mathbb{C}W = 0$), we have that $\mathrm{gr}(\mathbb{H}_k) = \Omega(V + V^*) \rtimes W$, where W acts trivially on $\bigwedge(V \oplus V^*)$.

Before we proceed to proving this theorem, we recall here some technical results about filtered modules and their tensor products.

Definition 4.2.5. Let R be a filtered ring with filtration $\{F^n R\}_{n \in \mathbb{Z}}$ and M a filtered R-module with filtration $\{F^n M\}_{n \in \mathbb{Z}}$. The module M is said to be free filtered (or *filt-free*) if it satisfies the following conditions:

(i) M is a free R-module with basis $\{m_i\}_{i \in I}$.

(ii) There exists a family of integers $\{n_i\}_{i\in I}$ such that $F^n M = \sum_{i\in I}(F^{n-n_i}R)\cdot m_i$ (It follows that $\deg m_i = n_i$ for all $i \in I$).

The set of pairs $\{(m_i, n_i) : i \in I\}$ is called a *filt-basis* of M.

Let R be a filtered ring, M a filtered left R-module and N a filtered right R-module. The tensor product $M \otimes_R N$ can be endowed with the filtration defined by $F^k(M \otimes_R N) = \sum_{i+j\leq k} F^i M \otimes F^j N$, where $F^i M \otimes F^j N$ denotes the abelian subgroup of $M \otimes N$ generated by all $m \otimes n$ with $m \in F^i M$ and $n \in F^j N$.

Lemma 4.2.6. *With above notation the natural graded morphism*

$$\phi : \mathrm{gr}(M) \otimes_{\mathrm{gr}(R)} \mathrm{gr}(N) \to \mathrm{gr}(M \otimes_R N)$$

is surjective. Moreover, if either M or N is filt-free, then ϕ is an isomorphism.

Proof. See [NVO82], p. 319.

□

Proof of Theorem 4.2.4. The first and last statements follow directly from Theorem 3.6.5 and 3.6.6 respectively, considering that $\mathbb{H}_k \simeq M_{2^n}(H_k)$ and the isomorphism

$$\mathbb{H}_0 = (\mathcal{D}(V) \rtimes W) \otimes Cl(V \oplus V^*) \simeq \mathcal{D}(\Omega V) \rtimes W$$

For (ii), notice that over the trivially graded $\mathbb{C}$, both H_k and $Cl(V \oplus V^*)$ are filt-free, and thus we have $\mathrm{gr}(\mathbb{H}_k) \simeq \mathrm{gr}(H_k) \otimes \mathrm{gr}(Cl(V \oplus V^*)) \simeq (\mathbb{C}[V \oplus V^*] \rtimes W) \otimes \bigwedge(V \oplus V^*) \simeq (\mathbb{C}[V \oplus V^*] \otimes \bigwedge(V \oplus V^*)) \rtimes W$, where W acts trivially on $\bigwedge(V \oplus V^*)$. A PBW basis for $\mathbb{H}_k$ can be formed by tensoring a PBW basis for H_k and a Clifford basis. The multiplication map is clearly an isomorphism of vector spaces, and is linear over ΩV. □

In the following section, we construct a Dunkl type representation for $\mathbb{H}_k$ that provides a Dunkl embedding of $\mathbb{H}_k$ into $\mathcal{D}(\Omega V_{reg}) \rtimes W$, and prove that the analogue of the classical localization lemma still holds in this context. More precisely, let $V_{reg} := V \setminus \{\delta = 0\}$, and so $\Omega V_{reg} = \Omega V[\delta^{-1}]$. Then we will prove:

Theorem. *There is an embedding $\mathbb{H}_k \hookrightarrow \mathcal{D}(\Omega V_{reg}) \rtimes W$. The set $\{\delta^n, n \geq 0\}$ is an Ore subset in $\mathbb{H}_k$, and the natural map $\mathbb{H}_k[\delta^{-1}] \to \mathcal{D}(\Omega V_{reg}) \rtimes W$ is an isomorphism of graded algebras.*

More interestingly, this embedding allows us to equip $\mathbb{H}_k$ with a dg algebra structure and we will prove:

Theorem 4.2.7. *The graded Cherednik algebra $\mathbb{H}_k = H_k \otimes Cl(V \oplus V^*)$ has a natural dg algebra structure where H_k sits in degree 0, $V \subset Cl(V \oplus V^*)$ in degree -1 and $V^* \subset Cl(V \oplus V^*)$ in degree 1. The differential on $\mathbb{H}_k$ is defined by:*

$$
\begin{aligned}
&d(s \otimes 1) = -s\alpha_s^\vee \otimes \alpha_s \\
&d(\xi \otimes 1) = 0 \\
&d(x \otimes 1) = 1 \otimes x - \sum_{s \in \Sigma} k_s \langle x, \alpha_s^\vee \rangle (s \otimes \alpha_s) \\
&d(1 \otimes \xi) = \xi \otimes 1 \\
&d(1 \otimes x) = 0
\end{aligned}
$$

where $\xi \in V$ and $x \in V^*$.

In fact, if $\{\xi_i\}$ and $\{x_i\}$ are dual bases of V and V^*, we will see that

$$d = \sum_i [\xi_i \otimes x_i, -]$$

where the commutator is the graded commutator inside $\mathbb{H}_k = H_k \otimes Cl(V \oplus V^*)$.

Proof. Neither the fact that d squares to zero, nor that it is a well defined derivation of $\mathbb{H}_k$ are obvious from the definition. We will prove both of these facts by constructing a Dunkl representation $\mathbb{H}_k \to \mathcal{D}(\Omega V_{reg}) \rtimes W$. The image of $\mathbb{H}_k$ inside $\mathcal{D}(\Omega V_{reg}) \rtimes W$ has a natural differential which we transport back to $\mathbb{H}_k$, see Theorem 4.4.1. $\square$

4.3 Dunkl representation

Recall that V is a finite dimensional vector space (over $\mathbb{C}$) and $W \subset \mathrm{GL}(V)$ is a finite reflection group acting on V. The action of W on V naturally defines an action of W on V^* and these actions extend to $A = \mathbb{C}[V] = \mathrm{Sym}(V^*)$ and $\mathbb{C}[V^*] = \mathrm{Sym}(V)$.

For a (pseudo-)reflection in W with hyperplane H, W_H is the cyclic subgroup of order n_H of elements in W that fix H. In this case, the restriction of the determinant character of W to W_H is cyclic generated by the determinant character. We write $\alpha_H \in V^*$ for a linear form s.t. $H = \ker(\alpha_H)$, and write Σ for the collection of reflection hyperplanes. Finally, let $\delta := \prod_{H\in\Sigma} \alpha_H$.

In this section, we construct a Dunkl type representation of $\mathbb{H}_k$ as equivariant differential operators.

Following [Dun89], we introduce and recall the construction of a deformed de Rham complex associated to a general complex reflection group (we will then restrict to the case when W is a Coxeter group in the later sections). We will need the following obvious statement to construct the embedding. Recall that

$V_{reg} := V \setminus \{\delta = 0\}$, and thus $\Omega V_{reg} = \Omega V[\delta^{-1}]$.

Proposition 4.3.1. *There are natural maps $\Omega V \to \Omega V_{reg}$, and $\underline{\mathrm{Der}}(\Omega V) \to \underline{\mathrm{Der}}(\Omega V_{reg})$ extending to natural maps $\Omega V[\delta^{-1}] \to \mathcal{D}(\Omega V_{reg})$, and $\mathcal{D}(\Omega V) \to \mathcal{D}(\Omega V_{reg})$*

Proof. The first map is just the localization map. The second one is a well known fact about localizing derivations [MR01]. Explicitly, if $D \in \underline{\mathrm{Der}}(\Omega V)$, then D extends to a derivation on $\Omega V_{reg} = \Omega V[\delta^{-1}]$ by the following formula $D(\frac{1}{\delta^m}\omega) = -\frac{m}{\delta^{m+1}}\omega + \frac{1}{\delta^m}D\omega$. The last statement follows from the fact that ΩV and $\underline{\mathrm{Der}}(\Omega V)$ generate $\mathcal{D}(\Omega V)$, see Theorem 3.5.8 above. □

Recall that we can extend the action of W on $\mathbb{C}[V]$ to a diagonal action on $\Omega V = \mathbb{C}[V] \otimes \bigwedge V^*$. This allows W to act on $\mathcal{D}(\Omega V)$ by conjugation. By extension, we also get an action of W on $\mathcal{D}(\Omega V_{reg}) = \mathcal{D}(\Omega V[\delta^{-1}]) = \mathcal{D}(V_{reg}) \otimes Cl(V \oplus V^*)$, and we form the crossed product algebras $\mathcal{D}(\Omega V) \rtimes W$ and $\mathcal{D}(\Omega V_{reg}) \rtimes W$.

4.3.1 Deformed de Rham differential

Let us start by recalling the following definitions. Given $H \in \mathcal{A}$ and $i \in \{0, 1, \cdots n_H - 1\}$ let

$$\begin{aligned} e_{H,i} &:= \frac{1}{n_H} \sum_{w \in W_H} \det{}_V(w)^{-i} w \in \mathbb{C}[W_H] \\ a_H &:= \sum_{i=1}^{n_H - 1} n_H k_{H,i} e_{H,i} \in \mathbb{C}[W_H] \end{aligned}$$

where $k_{H,i}$ are complex numbers with indices running over (W-orbits of) $H \in \mathcal{A}$ and for each H, $i \in \{1, \cdots, n_H - 1\}$. Define the following element

$$\omega_H := \alpha_H^{-1} d\alpha_H \in \Omega V_{reg}$$

Recall that $A = \mathbb{C}[V]$, and let us introduce the following definition.

Definition 4.3.2 ([DO03]). The operator $\Omega = \Omega(k) : A \to \Omega^1 A$ is defined as follows

$$\Omega := \sum_{H \in \mathcal{A}} a_H(\cdot)\omega_H$$

where $a_H \in \mathbb{C}[W_H] \subset \mathbb{C}[W]$ acts on A. In other words, for $f \in A$, we have $\Omega(f) = \sum_{H \in \Sigma} a_H(f)\omega_H$

Note 4.3.3. It looks as if Ω takes A to $(\Omega^1 A)[\delta^{-1}]$, but applying $a_H(\cdot)$ leads to something divisible by α_H, and the result is indeed in $\Omega^1 A$.

We then extend Ω to a degree 1 operator on the whole ΩA as follows. For $\omega \in \Omega A$, we define

$$\Omega(\omega) = \sum_H a_H \cdot (\omega_H \wedge \omega)$$

where a_H acts diagonally as explained earlier.

Note 4.3.4. ω_H is $\mathbb{C}[W_H]$ invariant and so $a_H \cdot \omega_H = 0$, but $a_H \cdot (\omega_H \wedge \omega)$ is non-zero (it is obviously not just $(a_H \cdot \omega_H) \wedge (a_H \cdot \omega)$).

In fact, if $f \in A$, $x_1, \cdots, x_n \in V^*$, and $g \in W_H$, then ${}^g x_i \in x_i + \mathbb{C}\alpha_H$, and so in this case:

$$\begin{aligned} \Omega(f dx_1 \wedge \cdots \wedge dx_n) &= \sum_H a_H(f)\omega_H \wedge dx_1 \wedge \cdots \wedge dx_n \\ &= \Omega(f) \wedge dx_1 \wedge \cdots \wedge dx_n \end{aligned}$$

Finally, we have the following definition.

Definition 4.3.5 ([DO03])**.** The deformed de Rham differential is the differential

$$\tilde{d} := d - \Omega \tag{4.3}$$

on ΩA, where d is the standard de Rham differential on ΩA.

The fact that $\tilde{d}$ is a differential, i.e. that $\tilde{d}^2 = 0$ is not obvious and is proved in [DO03]. Note that we chose here to flip the sign in front of Ω compared to the one in the original definition to better match the standard sign in the definition of the Cherednik algebras.

Note 4.3.6. The differential $\tilde{d}$ is *not* a (graded) derivation in general, see Example 4.3.7.

Still, if $f \in A$, $x_1, \cdots, x_n \in V^*$ (*not* A), then the following holds:

$$\begin{aligned}
\tilde{d}(f dx_1 \wedge \cdots \wedge dx_n) &= (d - \Omega) f dx_1 \wedge \cdots \wedge dx_n \\
&= df \wedge dx_1 \wedge \cdots \wedge dx_n - \Omega(f) \wedge dx_1 \wedge \cdots \wedge dx_n \\
&= \tilde{d} f \wedge dx_1 \wedge \cdots \wedge dx_n
\end{aligned}$$

4.3.2 Deformed Lie derivatives

Suppose now that W is a finite Coxeter group acting on its finite dimensional reflection representation V and let $\Sigma \subset W$ be the set of all (real) reflections in W. Then for each $s \in \Sigma$, let $\alpha_s \in V^*$ define the reflection hyperplane of s, and $\alpha_s^\vee \in V$, as defined earlier, be the (-1)-eigenvector such that $\langle \alpha_s, \alpha_s^\vee \rangle = 2$.

We recall here the following notation

$$
\begin{aligned}
H_s &=: \{\alpha_s = 0\} \\
n_s &= 2 \\
W_s &= \{1, s\} \\
\det{}_V|_{W_s} &= \text{sign} \\
e_{s,0} &= \frac{1}{2}(1+s) \\
e_{s,1} &= \frac{1}{2}(1-s) \\
a_s &= n_s k_{s,1} e_{s,1} = k_s(1-s) \\
\omega_s &= \alpha_s^{-1} d\alpha_s \\
\delta &:= \prod_s \alpha_s
\end{aligned}
$$

We get the following formula for Ω:

$$
\begin{aligned}
\Omega &= \sum_{s\in\Sigma} a_s(\cdot)\alpha_s^{-1} d\alpha_s \\
&= \sum_s k_s(1-s)(\cdot)\alpha_s^{-1} d\alpha_s
\end{aligned}
$$

where $s = s \otimes s$ acts on ΩA via the diagonal action.

We also have the following formula for $\tilde{d}$, for $x \in V^*$:

$$
\begin{aligned}
\tilde{d}x &= dx - \sum_s k_s(1-s)(x){\alpha_s}^{-1} d\alpha_s \\
&= dx - \sum_s k_s \langle x, \alpha_s^\vee \rangle d\alpha_s
\end{aligned}
$$

Example 4.3.7. Let $W = \mathbb{Z}/2$ act on $V = \mathbb{C}\xi$ and $V^* = \mathbb{C}x$ where $\langle x, \xi \rangle = 1$. Then with $\alpha_s = x$ and $\alpha_s^\vee = 2\xi$, $\tilde{d}(x^2) = 2xdx - k(1-s)(x^2)\frac{1}{x}dx = 2xdx$, while $2x\tilde{d}(x) = 2x(dx - 2kdx)$, and $\tilde{d}$ is not a derivation.

Definition 4.3.8. (Deformed Lie derivative) For $\xi \in V \subset \mathrm{Der}(A)$, we define the operator $\tilde{\mathcal{L}}_\xi \in \underline{\mathrm{End}}(\Omega A)$ by

$$\tilde{\mathcal{L}}_\xi := [\tilde{d}, i_\xi] = \mathcal{L}_\xi - [\Omega, i_\xi] =: \mathcal{L}_\xi - \pi_\xi \tag{4.4}$$

where $\mathcal{L}_\xi$ is the standard Lie derivative on ΩA with respect to ξ, and the commutator is a graded commutator.

Both $\mathcal{L}_\xi$ and $\tilde{\mathcal{L}}_\xi$ are graded endomorphisms of degree 0 on ΩA, but while $\mathcal{L}_\xi$ is a derivation, $\tilde{\mathcal{L}}_\xi$ is generally not as $\tilde{d}$ isn't.

Before we move on to the main definition of this section, we point out a couple of important identities.

Proposition 4.3.9. *We have the following identities*

(i) For $\xi \in V$*, we have* $\pi_\xi = \sum_s k_s \alpha_s^{-1} \langle \alpha_s, \xi \rangle (1 - (s \otimes 1))$*, and so*

$$\tilde{\mathcal{L}}_\xi = \mathcal{L}_\xi - \sum_s k_s \alpha_s^{-1} \langle \alpha_s, \xi \rangle (1 - (s \otimes 1)) \tag{4.5}$$

and thus

$$\tilde{\mathcal{L}}_\xi \in \mathcal{D}(\Omega V_{reg}) \rtimes W$$

(ii) Let $\{x_i\}$ *and* $\{\xi_i\}$ *be dual bases of* V^* *and* V *such that* $\langle x_i, \xi_j \rangle = \delta_{ij}$*. The differential* $\tilde{d}$ *can be split as follows:*

$$\tilde{d} = \sum_{i=1}^{n} m_{dx_i} \tilde{\mathcal{L}}_{\xi_i} \tag{4.6}$$

and thus $\tilde{d} \in \mathcal{D}(\Omega V_{reg}) \rtimes W$*.*

Proof. (i) The formula for $\tilde{\mathcal{L}}_\xi$ given in (4.5) is proved in the appendix. It follows from Proposition 4.3.1 (i) and Theorem 3.6.4 that $s \otimes 1 \in \mathcal{D}(\Omega V_{reg}) \rtimes W$, and thus so is $\tilde{\mathcal{L}}_\xi$.

(ii) We check the identity for 0-forms. For higher degree forms, the results follows from this and Note 4.3.6. For $i = 1, \ldots, n$ and $f \in A$, $\tilde{\mathcal{L}}_{\xi_i} f = \mathcal{L}_{\xi_i} f - \sum_s k_s(1 - s)(f){\alpha_s}^{-1}\langle \alpha_s, \xi_i \rangle$, and so

$$\begin{aligned}
\sum_i m_{dx_i} \tilde{\mathcal{L}}_{\xi_i} f &= \sum_i \mathcal{L}_{\xi_i} f dx_i - \sum_{s,i} k_s(1 - s)(f){\alpha_s}^{-1}\langle \alpha_s, \xi_i \rangle dx_i \\
&= \sum_i \mathcal{L}_{\xi_i} f dx_i - \sum_s k_s(1 - s)(f){\alpha_s}^{-1} \left(\sum_i \langle \alpha_s, \xi_i \rangle dx_i \right) \\
&= \sum_i \mathcal{L}_{\xi_i} f dx_i - \sum_s k_s(1 - s)(f){\alpha_s}^{-1} d\alpha_s \\
&= \tilde{d} f
\end{aligned}$$

□

4.3.3 The Dunkl embedding

A result very similar to the classical case still holds. More precisely:

Theorem 4.3.10. *(Dunkl embedding) The map $\mathbb{H}_k \to \mathcal{D}(\Omega V_{reg}) \rtimes W$ given on generators by*

$$\begin{aligned}
s \otimes 1 &\mapsto \frac{1}{2} \left(i_{\alpha_s^\vee} m_{d\alpha_s} - m_{d\alpha_s} i_{\alpha_s^\vee} \right) s \\
\xi \otimes 1 &\mapsto \tilde{\mathcal{L}}_\xi \\
x \otimes 1 &\mapsto m_x \\
1 \otimes \xi &\mapsto i_\xi \\
1 \otimes x &\mapsto m_{dx}
\end{aligned}$$

is a morphism of graded algebras and yields an embedding of $\mathbb{H}_k$ *onto its image* $\mathcal{H}_k \subset \mathcal{D}(\Omega V_{reg}) \rtimes W$*. Indeed, the following relations hold:*

$$
\begin{aligned}
\hat{s}^2 &= 1 \\
{i_\xi}^2 &= 0 \\
{m_{dx}}^2 &= 0 \\
[\hat{s}, i_\xi] &= 0 \\
[\hat{s}, m_x] &= -\langle x, \alpha_s^\vee \rangle \alpha_s \hat{s} \quad (\hat{s} m_x \hat{s} = m_{^s x}) \\
[\hat{s}, m_{dx}] &= 0 \\
\left[\hat{s}, \tilde{\mathcal{L}}_\xi\right] &= \hat{s} \langle \alpha_s, \xi \rangle \tilde{\mathcal{L}}_{\alpha_s^\vee} \quad (\hat{s} \tilde{\mathcal{L}}_\xi \hat{s} = \tilde{\mathcal{L}}_{^s\xi}) \\
[i_\xi, m_x] &= 0 \\
[m_z, m_x] &= 0 \\
[m_{dz}, m_{dx}] &= 0 \\
[i_\xi, i_\eta] &= 0 \\
[i_\xi, m_{dx}] &= \langle x, \xi \rangle \\
\left[\tilde{\mathcal{L}}_\xi, i_\eta\right] &= i_{[\xi,\eta]} = 0 \\
\left[\tilde{\mathcal{L}}_\xi, m_x\right] &= \left(\langle x, \xi \rangle - \sum_s k_s \langle x, \alpha_s^\vee \rangle \langle \alpha_s, \xi \rangle \hat{s} \right) \\
\left[\tilde{\mathcal{L}}_\xi, m_{dx}\right] &= 0 \\
\left[\tilde{\mathcal{L}}_\xi, \tilde{\mathcal{L}}_\eta\right] &= 0
\end{aligned}
$$

for $x, z \in V^*$*,* $\xi, \eta \in V$*, and* $\hat{s} := \frac{1}{2}\left(i_{\alpha_s^\vee} m_{d\alpha_s} - m_{d\alpha_s} i_{\alpha_s^\vee}\right) s$ *(all commutators are graded).*

Proof. The computations of the commutator relations are provided in the appendix and show that this map is well defined. It is clear that this is an embedding as it is simply $\mathbb{H}_k \mapsto (\mathcal{D}(V_{reg}) \rtimes W) \otimes 1$, $Cl(V \oplus V^*) \mapsto 1 \otimes Cl(V \oplus V^*)$ inside $\mathcal{D}(\Omega V_{reg}) \rtimes W \simeq (\mathcal{D}(V_{reg}) \rtimes W) \otimes Cl(V \oplus V^*)$, and $H_k \hookrightarrow \mathcal{D}(V_{reg})$ is an embedding. $\square$

We then conclude this section with the transpose of a classical result:

Theorem 4.3.11. *Let $\mathbb{H}_{reg} = \mathbb{H}_k[\delta^{-1}]$ denote the localization of $\mathbb{H}_k$ at the Ore subset $\{\delta^k\}_{\{k \in \mathbb{N}\}}$. Then the Dunkl embedding induces a map $\mathbb{H}_{reg} \to \mathcal{D}(\Omega V_{reg}) \rtimes W$ which is an isomorphism of algebras.*

Proof. Follows from the statement in the classical case, see [BEG03]. Indeed, the only poles of the Dunkl operators are still at the reflection hyperplanes. By adding the inverse of δ to ΩV, we can now produce all the Lie derivatives and hence generate all of $\mathcal{D}(\Omega V_{reg}) \rtimes W$. $\square$

4.4 The differential structure

More than just a graded algebra structure, we have in fact a natural differential on $\mathcal{H}_k$:

Theorem 4.4.1. *The graded algebra $\mathcal{H}_k$ has a natural dg structure. More explicitly, one defines a differential on $\mathcal{H}_k$ by taking the graded commutator with $\tilde{d}$, i.e.*

$$
\begin{aligned}
[\tilde{d}, \hat{s}] &= -\hat{s} m_{d\alpha_s} \tilde{\mathcal{L}}_{\alpha_s^\vee} \\
[\tilde{d}, m_x] &= m_{dx} - \sum_s k_s \langle x, \alpha_s^\vee \rangle m_{d\alpha_s} \hat{s} \\
[\tilde{d}, m_{dx}] &= 0 \\
[\tilde{d}, i_\xi] &= \tilde{\mathcal{L}}_\xi \\
[\tilde{d}, \tilde{\mathcal{L}}_\xi] &= 0
\end{aligned}
$$

for $x \in V^*$ and $\xi \in V$, and where $\hat{s} := \frac{1}{2}\left(i_{\alpha_s^\vee} m_{d\alpha_s} - m_{d\alpha_s} i_{\alpha_s^\vee}\right) s$

This dga structure transports back to $\mathbb{H}_k$ and we get that the graded Cherednik algebra has a dg structure where the differential is given on generators as shown in Theorem 4.2.7.

Also, if $\{\xi_i\}$ and $\{x_i\}$ are dual bases of V and V^*, then it follows from Proposition 4.3.9 (iii) that the differential d on $\mathbb{H}_k$ satisfies

$$d = \sum_i [\xi_i \otimes x_i, -]$$

where the commutator is the graded commutator inside $\mathbb{H}_k$.

Before we prove theorem 4.4.1, we need the following lemma:

Lemma 4.4.2. *The following equalities hold:*

(i) $\Omega({}^s f) = s\Omega f$, *for* $f \in A$, *and* $s \in W$ *a reflection.*

(ii) $\Omega(fg) = f\Omega(g) + g\Omega(f) - \sum_s k_s(g - {}^s g)(f - {}^s f){\alpha_s}^{-1} d\alpha_s$, *for* $f, g \in A$.

Proof. (i) For two reflections s and σ, $\sigma s \sigma$ is also a reflection and its fixed hyperplane is the image under s of that of σ. This means that $\alpha_{\sigma s \sigma} \in \mathbb{C} \cdot {}^\sigma \alpha_s$. We then have:

$$\begin{aligned}
\Omega(\sigma f) &= \sum_s k_s(1 - s)(\sigma f){\alpha_s}^{-1} d\alpha_s \\
&= \sum_s {k_s}^\sigma((1 - \sigma s \sigma) f)({}^\sigma \alpha_{\sigma s \sigma}^{-1}) d({}^\sigma \alpha_{\sigma s \sigma}) \\
&= \sum_s k_s \sigma\left((1 - \sigma s \sigma) f \alpha_{\sigma s \sigma}^{-1} d(\sigma \alpha_{\sigma s \sigma})\right) \\
&= \sigma\Omega(f)
\end{aligned}$$

(ii) Let now $f, g \in A$ and consider the following equalities:

$$\begin{aligned}
\Omega(fg) &= \sum_s k_s(1-s)(fg){\alpha_s}^{-1}d\alpha_s \\
&= \sum_s k_s(fg - {}^sf{}^sg){\alpha_s}^{-1}d\alpha_s \\
&= \sum_s k_s(fg - f{}^sg + f{}^sg - {}^sf{}^sg){\alpha_s}^{-1}d\alpha_s \\
&= \sum_s k_s(f(g - {}^sg) + (f - {}^sf){}^sg){\alpha_s}^{-1}d\alpha_s \\
&= f\Omega(g) + \sum_s k_s{}^sg(f - {}^sf){\alpha_s}^{-1}d\alpha_s \\
&= f\Omega(g) + \sum_s k_s(g - (g - {}^sg))(f - {}^sf){\alpha_s}^{-1}d\alpha_s \\
&= f\Omega(g) + g\Omega(f) - \sum_s k_s(g - {}^sg)(f - {}^sf){\alpha_s}^{-1}d\alpha_s
\end{aligned}$$

□

Proof of theorem 4.4.1. (i) First, because $\tilde{d}$ has odd degree, it is clear that $[\tilde{d}, [\tilde{d}, -]] = 0$ and thus $[\tilde{d}, -]$ is a differential on $\underline{\mathrm{End}}(\Omega V_{reg})$. We compute the action of this differential on the generators of $\mathcal{H}_k$. This will show that $\tilde{d}$ preserves $\mathcal{H}_k$ and prove the equalities listed.

(ii) By definition $[\tilde{d}, i_\xi] = \tilde{\mathcal{L}}_\xi$.

(iii) The non trivial identify $\tilde{d}^2 = 0$ (proved in [DO03]) translates to:

$$[\tilde{d}, \tilde{d}] = \tilde{d}\tilde{d} - (-1)^1\tilde{d}\tilde{d} = 2\tilde{d}\tilde{d} = 0$$

which yields

$$\begin{aligned}
[\tilde{d}, \tilde{\mathcal{L}}_\xi] &= [\tilde{d}, [\tilde{d}, i_\xi]] \\
&= [[\tilde{d}, \tilde{d}], i_\xi] + (-1)^1[\tilde{d}, [\tilde{d}, i_\xi]] \\
&= -[\tilde{d}, [\tilde{d}, i_\xi]] = -[\tilde{d}, \tilde{\mathcal{L}}_\xi] = 0
\end{aligned}$$

(iv) For $f \in A$, and $dz = dz_1 \wedge \cdots \wedge dz_k$ with $z_i \in V^*$, we have

$$\begin{aligned}
[\tilde{d}, m_{dx}](f dz) &= \tilde{d}(f dx \wedge dz) - (-1)^1 dx \wedge \tilde{d}(f dz) \\
&= \tilde{d}(f) \wedge dx \wedge dz + dx \wedge \tilde{d}(f) \wedge dz \\
&= \tilde{d}(d) \wedge dx \wedge dz - \tilde{d}(f) \wedge dx \wedge dz \\
&= 0
\end{aligned}$$

(v) with notation as above, we have

$$\begin{aligned}
[\tilde{d}, m_x](f dz) &= \tilde{d}(x f dz) - (-1)^{1 \cdot 0} x \tilde{d}(f dz) \\
&= (\tilde{d}(xf) - x\tilde{d}(f)) \wedge dz \\
&= (d(xf) - \Omega(xf) - x(df - \Omega(f))) \wedge dz \\
&= \left((dx) f - f\Omega(x) + \sum_s k_s (x - {}^s x)(f - {}^s f) \alpha^{-1} d\alpha_s \right) \wedge dz \\
&= \tilde{d}x \wedge f dz + \sum_s k_s (x - {}^s x)(f - {}^s f) \alpha^{-1} d\alpha_s \wedge dz \\
&= \tilde{d}x \wedge f dz + \sum_s k_s \langle x, \alpha_s^\vee \rangle (f - {}^s f) d\alpha_s \wedge dz \\
&= \left(m_{\tilde{d}x} + \sum_s k_s \langle x, \alpha_s^\vee \rangle m_{d\alpha_s} (1 - \hat{s}) \right) (f dz) \\
&= \left(m_{dx} - \sum_s k_s \langle x, \alpha_s^\vee \rangle m_{d\alpha_s} + \sum_s k_s \langle x, \alpha_s^\vee \rangle m_{d\alpha_s} (1 - \hat{s}) \right) (f dz) \\
&= \left(m_{dx} - \sum_s k_s \langle x, \alpha_s^\vee \rangle m_{d\alpha_s} \hat{s} \right) (f dz)
\end{aligned}$$

Alternatively, one can also compute

$$\begin{aligned}
[\tilde{d}, m_x] &= [\sum_i m_{dx_i} \tilde{\mathcal{L}}_{\xi_i}, m_x] \\
&= \sum_i [m_{dx_i} \tilde{\mathcal{L}}_{\xi_i}, m_x] \\
&= \sum_i m_{dx_i} [\tilde{\mathcal{L}}_{\xi_i}, m_x] + (-1)^0 [m_{dx_i}, m_x] \tilde{\mathcal{L}}_{\xi_i} \\
&= \sum_i m_{dx_i} \left(\langle x, \xi_i \rangle - \sum_s k_s \langle x, \alpha_s^\vee \rangle \langle \alpha_s, \xi_i \rangle \hat{s} \right) \\
&= m_{d \sum_i \langle x, \xi_i \rangle x_i} - \sum_s k_s \langle x, \alpha_s^\vee \rangle m_{d \sum_i \langle \alpha_s, \xi_i \rangle x_i} \hat{s} \\
&= m_{dx} - \sum_s k_s \langle x, \alpha_s^\vee \rangle m_{d\alpha_s} \hat{s}
\end{aligned}$$

(vi) Finally, we compute $[\tilde{d}, \hat{s}]$. The following computation for 1-forms $f dz$, $z \in V^*$, generalizes to arbitrary forms:

$$\begin{aligned}
[\tilde{d}, \hat{s}](f dz) &= \tilde{d}({}^s f dz) - (-1)^{1 \cdot 0} \hat{s}(\tilde{d} f \wedge dz) \\
&= \tilde{d}({}^s f) \wedge dz - \hat{s}(\tilde{d} f) \wedge dz \\
&= (d({}^s f) + \Omega({}^s f) - \hat{s}(df + \Omega f)) \wedge dz \\
&= (d({}^s f) + s\Omega(f) - \hat{s} df - \hat{s} \Omega f) \wedge dz \\
&= (s - \hat{s})(df + \Omega(f)) \wedge dz \\
&= (s - \hat{s})(\tilde{d} f) \wedge dz \\
&= (s \tilde{d} f) dz - (\hat{s} \tilde{d} f) dz \\
&= (s \tilde{d} f) d({}^s z - {}^s z + z) - \hat{s} \tilde{d}(f dz) \\
&= (s \tilde{d} f) d({}^s z) - (s \tilde{d} f) d({}^s z - z) - \hat{s} \tilde{d}(f dz) \\
&= s(\tilde{d} f dz) - (s \tilde{d} f) d({}^s z - z) - \hat{s} \tilde{d}(f dz) \\
&= (s - \hat{s}) \tilde{d}(f dz) - (s \tilde{d} f) d({}^s z - z) \\
&= (s - \hat{s}) \tilde{d}(f dz) + (s \tilde{d} f) \langle z, \alpha_s^\vee \rangle d\alpha_s
\end{aligned}$$

$$
\begin{aligned}
&= \left(-m_{d\alpha_s} s i_{\alpha_s^\vee} \tilde{d} - m_{d\alpha_s} s \tilde{d} i_{\alpha_s^\vee}\right)(f dz) \\
&= (-m_{d\alpha_s} s \tilde{\mathcal{L}}_{\alpha_s^\vee})(f dz) \\
&= (-m_{d\alpha_s} (s - \hat{s} + \hat{s}) \tilde{\mathcal{L}}_{\alpha_s^\vee})(f dz) \\
&= (-m_{d\alpha_s} \hat{s} \tilde{\mathcal{L}}_{\alpha_s^\vee})(f dz)
\end{aligned}
$$

Alternatively, we have

$$
\begin{aligned}
[\tilde{d}, \hat{s}] &= [\sum_i m_{dxi} \tilde{\mathcal{L}}_\xi, \hat{s}] \\
&= \sum_i [m_{dxi} \tilde{\mathcal{L}}_\xi, \hat{s}] \\
&= \sum_i m_{dxi} [\tilde{\mathcal{L}}_\xi, \hat{s}] + (-1)[m_{dxi}, \hat{s}] \tilde{\mathcal{L}}_\xi \\
&= -\sum_i m_{dxi} [\hat{s}, \tilde{\mathcal{L}}_{\xi_i}] \\
&= -\sum_i m_{dxi} \hat{s} \langle \alpha_s, \xi_i \rangle \tilde{\mathcal{L}}_{\alpha_s^\vee} \\
&= -\hat{s} m_{d \sum_i \langle \alpha_s, \xi_i \rangle x_i} \tilde{\mathcal{L}}_{\alpha_s^\vee} \\
&= -\hat{s} m_{d\alpha_s} \tilde{\mathcal{L}}_{\alpha_s^\vee}
\end{aligned}
$$

□

4.5 Cohomology

4.5.1 Singular polynomials

We now investigate the cohomology of $\mathbb{H}_k$. We formulate a conjecture for the cohomology groups and prove that it holds in the rank 1 case. In the rest of this section, we will often simply write $H := H_k(W)$ for the standard rational Cherdenik algebra of the Coxeter group W acting on V.

Let $\mathrm{Irr}(W)$ be the set of isomorphism classes of irreducible representations of W and let τ be a $\mathbb{C}W$-module. We can make τ into a $\mathbb{C}[V^*] \rtimes W$-module by making $\mathbb{C}[V^*]$ act trivially on τ. We can then form the following left H-module

$$M(\tau) := \mathrm{Ind}^{H}_{\mathbb{C}[V^*]\rtimes W}\,\tau = H_k \otimes_{\mathbb{C}[V^*]\rtimes W} \tau \tag{4.7}$$

When τ is irreducible, $M(\tau)$ is the *standard module of type* τ (see [BC11]). Notice that since the right multiplication by W preserves $H\xi$, $H/H\xi$ is in fact a right $\mathbb{C}W$-module, and since $\mathbb{C}[V^*]$ acts trivially on τ, it is clear that

$$H \otimes_{\mathbb{C}[V^*]\rtimes W} \tau \cong (H/H\xi) \otimes_{\mathbb{C}W} \tau \tag{4.8}$$

Let us describe $M(\tau)$ more explicitly. As a module over H,

$$M(\tau) \cong (\mathbb{C}[V] \otimes \mathbb{C}[V^*] \otimes \mathbb{C}W) \otimes_{\mathbb{C}[V^*]\rtimes W} \tau$$

and thus it is isomorphic to $C[V] \otimes \tau$ where the action of $W, \mathbb{C}[V]$ and $\mathbb{C}[V^*]$ are as follows

$$\begin{aligned} x(f \otimes v) &= xf \otimes v \\ s(f \otimes v) &= {}^s f \otimes {}^s v \\ \xi(f \otimes v) &= \partial_\xi f \otimes v + \sum_s k_s \frac{\langle \alpha_s, \xi \rangle}{\alpha_s}(f - {}^s f) \otimes sv \end{aligned} \tag{4.9}$$

for $f \in \mathbb{C}[V], v \in \tau$ and $\xi \in V$. Note that (4.9) simply says that inside $M(\tau)$, the following equalities hold:

$$(\xi f) \otimes v = [\xi, f] \otimes v = \partial_\xi f \otimes v + \sum_s k_s \frac{\langle \alpha_s, \xi \rangle}{\alpha_s}(f - {}^s f) \otimes sv$$

which can be checked easily on monomials.

Define now $\mathfrak{v}$ to be the commutative Lie algebra on the vector space V. Then $\mathfrak{v} = V \subset H_k$ acts on $M(\tau)$ and hence on $M(\tau) \otimes \tau^*$, and we can form $M(\tau)^{\mathfrak{v}} \otimes \tau^*$, which is the subset of elements in $M(\tau)$ which are killed by the action of *all* the Dunkl operators. We form the following conjecture.

Conjecture 4.5.1. The cohomology of $\mathbb{H}_k$ is

$$H^0(\mathbb{H}_k) \cong \bigoplus_{\tau \in \mathrm{Irr}(W)} M(\tau)^{\mathfrak{v}} \otimes \tau^* \tag{4.10}$$

$$H^i(\mathbb{H}_k) = 0, \ i \neq 0, \ k \text{ integral} \tag{4.11}$$

Let us interpret this result. When τ is the trivial representation, the space $M(\mathrm{triv})^{\mathfrak{v}}$ is simply $\mathcal{S} \otimes \tau$, where $\mathcal{S}$ is the space of *singular polynomials* of the group W acting on $\mathbb{C}[V]$ ([DDJO94]). For regular values of k, the common kernel of all Dunkl operators is trivial (i.e. contains only constants), in which case $M(\mathrm{triv})^{\mathfrak{v}} = \mathrm{triv}$. For a general representation τ, we have a similar notion of singular polynomials, where the Dunkl operators act in "representation τ", in other words as given by (4.9). If k is integral then it is known that k is regular in *all* representations ([BC11]), in other words the common kernel of all Dunkl operators is trivial in all representations. We thus have the following result.

Theorem 4.5.2. *If k is regular, then*

$$\bigoplus_{\tau \in \mathrm{Irr}(W)} M(\tau)^{\mathfrak{v}} \otimes \tau^* = \mathbb{C}W \tag{4.12}$$

in which case, assuming Conjecture 4.5.1 holds, we have

$$H^0(\mathbb{H}_k) \cong \mathbb{C}W \tag{4.13}$$

Proof. This follows from the isomorphism $\mathbb{C}W \cong \sum_{\tau \in \mathrm{Irr}(W)} \tau \otimes \tau^*$. □

We now move on to to proving Conjecture 4.5.1 in the rank 1 case. Doing so, we will obtain the following result:

Theorem 4.5.3. *The cohomology of* $\mathbb{H}_k(\mathbb{Z}/2)$ *is given by*

$$H^{-1}(\mathbb{H}_k) = 0 \quad \textit{for all } k \tag{4.14}$$

$$\begin{aligned} H^0(\mathbb{H}_k) &\cong (H/H\xi)^{\mathrm{ad}_\xi} \cong M(triv)^\xi \otimes triv \bigoplus M(sign)^\xi \otimes sign \\ &\cong \mathbb{C}W, \quad \textit{if } k \notin \mathbb{Z} + \frac{1}{2} \end{aligned} \tag{4.15}$$

$$H^1(\mathbb{H}_k) = 0, \quad \textit{if } k \notin \mathbb{Z} + \frac{1}{2} \tag{4.16}$$

4.5.2 The rank 1 case

Let W be the group $\mathbb{Z}/2 = \{1, s\}$ with $s^2 = 1$, acting on the one dimensional vector space $V = \mathbb{C}\xi$ by reflection around the origin, i.e. $s\xi = -\xi$, and write $V^* = \mathbb{C}x$, where $\langle x, \xi \rangle = 1$. We then have $\mathbb{C}[V] = \mathbb{C}[x]$, $\mathbb{C}[V^*] = \mathbb{C}[\xi]$, and $\Omega V = \mathbb{C}[x, dx]/\{dx^2 = 0\}$ is the exterior algebra of (commutative) forms on V.

Recall that each element $\xi \in V$ defines an element $\xi \in \mathrm{Der}(A)$, written ξ as well, by extending the pairing $\langle -, \xi \rangle$. In fact such derivations generate $\mathrm{Der}(A)$ over A. Explicitly, if $\xi \in V$ and $x \in V^*$, then $i_\xi dx = \langle x, \xi \rangle$, and for the commutator as derivations of A, $[\xi, x] = \langle x, \xi \rangle$. Finally, the set $\Sigma \subset W$ of all reflections in W is just $\Sigma = \{s\}$, and we choose $\alpha_s = x \in V^*$ to define the reflection hyperplane of s, and so $\alpha_s^\vee = 2\xi \in V$ is the (-1)-eigenvector such that $\langle \alpha_s, \alpha_s^\vee \rangle = 2$.

The algebra $\mathbb{H}_k$ is then by definition $H_k \otimes Cl(V \oplus V^*)$, where

$$H_k := \mathbb{C}\langle \xi, x, s \rangle / \langle s^2 = 1,\ s\xi s = -\xi,\ sxs = -x,\ \xi x - x\xi = 1 - 2ks \rangle \tag{4.17}$$

$$Cl(V \oplus V^*) := \mathbb{C}\langle \xi, x \rangle / \langle \xi^2 = x^2 = 0,\ \xi x + x\xi = 1 \rangle$$

and the Dunkl embedding $\mathbb{H}_k \to \mathcal{D}(\Omega V_{reg}) \rtimes \mathbb{Z}/2$ is given here by

$$\begin{aligned}
&s \otimes 1 \mapsto s\,(i_\xi m_{dx} - m_{dx} i_\xi) =: \hat{s} \\
&\xi \otimes 1 \mapsto L_\xi - \frac{k}{x}(1 - \hat{s}) \\
&x \otimes 1 \mapsto x \\
&1 \otimes \xi \mapsto i_\xi \\
&1 \otimes x \mapsto m_{dx}
\end{aligned}$$

Finally, recall the definition of the grading on $\mathbb{H}_k$, given by

$$\deg(1 \otimes x) = 1, \ \deg(1 \otimes \xi) = -1, \ \deg(H_k \otimes 1) = 0$$

The differential is now simply

$$d = [\xi \otimes x, -] \tag{4.18}$$

where the commutator is the graded commutator inside $\mathbb{H}_k$. Its action on the generators of $\mathbb{H}_k$ is given by

$$\begin{aligned}
&d(s \otimes 1) = -2s\xi \otimes x \\
&d(\xi \otimes 1) = 0 \\
&d(x \otimes 1) = (1 - 2ks) \otimes x \\
&d(1 \otimes \xi) = \xi \otimes 1 \\
&d(1 \otimes x) = 0
\end{aligned}$$

Proposition 4.5.4. *The cohomology of $\mathbb{H}_k$ is isomorphic to the cohomology of the following complex of vector spaces:*

$$\begin{array}{ccccccc}
0 \longrightarrow & H_k & \longrightarrow & H_k \oplus H_k & \longrightarrow & H_k & \longrightarrow 0 \\
& \alpha & \longmapsto & (\alpha\xi, \xi\alpha) & & & \\
& & & (\alpha, \beta) & \longmapsto & \xi\alpha - \beta\xi &
\end{array} \tag{4.19}$$

with $H_k \oplus H_k$ being in degree 0, and H_k in degree -1 on the left, and degree +1 on the right.

Proof. Inside $Cl(V \oplus V^*)$, $\xi^2 = x^2 = 0$ and $\xi x = 1 - x\xi$, and thus

$$\xi x \xi = \xi(1 - \xi x) = \xi, \quad x\xi x = x(1 - x\xi) = x \tag{4.20}$$

and so on. It is then clear that in degree zero we have ${\mathbb{H}_k}^0 = (H_k \otimes \xi x) \oplus (H_k \otimes x\xi)$, where the direct sum splitting is as vector spaces (or in fact as H_k-modules). In degree -1 we have ${\mathbb{H}_k}^{-1} = (H_k \otimes \xi)$, while in degree 1, ${\mathbb{H}_k}^1 = (H_k \otimes x)$. Putting it all in a diagram we have:

$$\mathbb{H}_k{}^\bullet = \Big[\, 0 \longrightarrow H_k \otimes \xi \longrightarrow (H_k \otimes \xi x) \oplus (H_k \otimes x\xi) \longrightarrow H_k \otimes x \longrightarrow 0 \,\Big]$$

where one has to be careful and remember that this is not a diagram of H_k-modules, as the differential d is not H_k linear.

Let us compute the action of d in this splitting. For $\alpha \in H_k$ we have

$$\begin{aligned}
d(\alpha \otimes \xi x) &= [\xi \otimes x, \alpha \otimes \xi x] \\
&= \xi\alpha \otimes x\xi x - \alpha\xi \otimes \xi x^2 \\
&= \xi\alpha \otimes x\xi x \\
d(\alpha \otimes x\xi) &= [\xi \otimes x, \alpha \otimes x\xi] \\
&= \xi\alpha \otimes x^2\xi - \alpha\xi \otimes x\xi x \\
&= -\alpha\xi \otimes x \\
d(\alpha \otimes \xi) &= [\xi \otimes x, \alpha \otimes \xi] \\
&= \xi\alpha \otimes x\xi + \alpha\xi \otimes \xi x
\end{aligned}$$

Making the natural identifications $H_k \otimes x \simeq H_k$, $(H_k \otimes \xi x) \oplus (H_k \otimes x\xi) \simeq H_k \oplus H_k$ and $H_k \otimes x \simeq H_k$, we see that the cohomology of $\mathbb{H}_k$ is the cohomology of the complex (4.19). $\square$

Now while the differential is obvious in the previous diagram, it is not at all clear how the multiplication in $\mathbb{H}_k$ works in this splitting. To see the algebra structure on ${\mathbb{H}_k}^0$ and $H^0(\mathbb{H}_k)$, notice that in fact ${\mathbb{H}_k}^0$ is actually isomorphic to $H_k \times H_k$ as an algebra. Indeed $(\alpha_1 x\xi + \beta_1 \xi x)(\alpha_2 x\xi + \beta_2 \xi x) = \alpha_1\alpha_2 x\xi + \beta_1\beta_2 \xi x$, as for example $x\xi x\xi = x(1 - x\xi)\xi = x\xi$ and $x\xi\xi x = 0$. This should not come as a surprise. Indeed, recall that $Cl(V \oplus V^*)$, as an algebra, is isomorphic to $M_2(\mathbb{C})$ the algebra of 2x2 matrices over $\mathbb{C}$, where the isomorphism comes from identifying

$$\xi \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad x \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Thus, as an ungraded algebra, $H \otimes Cl(V \oplus V^*)$ is isomorphic to the algebra $M_2(H)$ of 2 by 2 matrices with coefficients in H. Now, taking into account the grading, the diagonal matrices in $M_2(H)$ have degree 0, while the (strictly) upper triangular ones have degree 1 and the lower triangular have degree -1. With this in mind, the complex given in (4.19) can then be rewritten as:

$$\begin{array}{ccccccccc}
0 \longrightarrow & \begin{pmatrix} 0 & 0 \\ H & 0 \end{pmatrix} & \longrightarrow & \begin{pmatrix} H & 0 \\ 0 & H \end{pmatrix} & \longrightarrow & \begin{pmatrix} 0 & H \\ 0 & 0 \end{pmatrix} & \longrightarrow 0 \\
& \begin{pmatrix} 0 & 0 \\ \alpha & 0 \end{pmatrix} & \longmapsto & \begin{pmatrix} \alpha\xi & 0 \\ 0 & \xi\alpha \end{pmatrix} & & & \\
& & & \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} & \longmapsto & \begin{pmatrix} 0 & \xi\alpha - \beta\xi \\ 0 & 0 \end{pmatrix} &
\end{array}$$

We are now ready to start computing the cohomology of $\mathbb{H}_k$. We start with the cohomology in degree -1 and degree 1, as computing the cohomology in degree 0 will require more work.

Theorem 4.5.5. *We have the following isomorphisms:*

$$H^{-1}(\mathbb{H}_k) = 0$$
$$H^1(\mathbb{H}_k) \cong 0, \quad \textit{if } k \notin \mathbb{Z} + \frac{1}{2}$$

Proof. The cohomology in degree -1 is clear, as ξ is torsion-less in H. Now, for the cohomology in degree 1, it is clear that $H^1(\mathbb{H}_k) \cong H/(H\xi + \xi H)$ as a vector space. Recall that, still as a vector space, H has the following decomposition

$$H = \bigoplus_{m \geq 0} (\mathbb{C}[\xi] \oplus \mathbb{C}[\xi]s)x^m \tag{4.21}$$

When taking the quotient by the vector space $H\xi + \xi H$, it is clear that we get a quotient of

$$\bigoplus_{m \geq 0} (\mathbb{C} \oplus \mathbb{C}s)x^m \tag{4.22}$$

An easy computation shows that

$$[\xi, x^{2m}] = 2mx^{2m-1}, \ m \geq 1$$
$$[\xi, x^{2m+1}] = (2m + 1 - 2ks)x^{2m}, \ m \geq 0$$

where $\lambda(2m+1) := (2m+1-2ks)$ is invertible in H if $k \notin \mathbb{Z} + \frac{1}{2}$. Indeed, in that case $(2m + 1)^2 - 4k^2 \neq 0$, as the difference between an odd number and an even one, and we have

$$(2m + 1 - 2ks)\frac{2m + 1 + 2ks}{(2m + 1)^2 - 4k^2} = 1$$

Thus in the quotient, $x^{2m-1} = \frac{1}{2m}[\xi, x^{2m}] \equiv 0$ for $m \geq 1$, and $x^{2m} = \lambda(2m + 1)^{-1}[\xi, x^{2m+1}] \equiv 0$ for $m \geq 0$. Finally, since $p(x) \equiv 0$ implies that $(a + bs)p(x) \equiv 0$ for all a, b, $H^1(\mathbb{H}_k) = 0$ if $k \notin \mathbb{Z} + \frac{1}{2}$. □

We are now ready to move on to computing the cohomology in degree 0. Recall that given a ring R and a left ideal $I \subset R$, the *idealizer* $\mathbb{I}(I) \subseteq R$ of I in R is the largest subring of R in which I is a two-sided ideal. In other words

$$\mathbb{I}(I) := \{r \in R : Ir \subset I\} \tag{4.23}$$

and I is a two-sided ideal of $\mathbb{I}(I)$, so that the quotient $\mathbb{I}(I)/I$ inherits a ring structure from R.

Proposition 4.5.6. *Let $H\xi$ denote the left ideal generated by ξ inside of H. Then*

$$H^0(\mathbb{H}_k) = \mathbb{I}(H\xi)/H\xi \tag{4.24}$$

Proof. Let $\pi : H_k \otimes H_k \twoheadrightarrow H_k$ be the projection on the first factor. Then we claim that $\pi|_{\ker d^0}$ is injective. Indeed, if $(\alpha, \beta) \in \ker d^0$ is such that $\pi(\alpha, \beta) = 0$, then by definition $\alpha = 0$, and since $\xi\alpha = \beta\xi$, $\beta\xi = 0$ and hence $\beta = 0$ since ξ is torsion free in H. Next, we show that $\operatorname{Im}(\pi|_{\ker d^0}) = \mathbb{I}(H\xi)$. Indeed, the image consists of $\alpha \in H$ such that there exists $\beta \in H$ for which $(\alpha, \beta) \in \ker d^0$, in other words

$$\operatorname{Im}(\pi|_{\ker d^0}) = \{\alpha \in H : \xi\alpha \in H\xi\}$$

which is indeed $\mathbb{I}(H\xi)$ by definition.

Consider now the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Im} d^{-1} \arrow[r] \arrow[d, "\cong"'] & \ker d^0 \arrow[r] \arrow[d, "\cong"', "\pi"] & H^0(\mathbb{H}_k) \arrow[r] & 0 \\
0 \arrow[r] & H\xi \arrow[r] & \mathbb{I}(H\xi) \arrow[r] & \mathbb{I}(H\xi)/H\xi \arrow[r] & 0
\end{tikzcd}$$

Then π descends to a map $H^0(\mathbb{H}_k) \to \mathbb{I}(H\xi)/H\xi$ and the Snake lemma shows that the induced map is in fact an isomorphism. $\square$

Next, we identify $\mathbb{I}(H\xi)/H\xi$. Write ad_ξ for the commutator $[\xi, -]$ inside H. Then we have

Proposition 4.5.7. *The action of* ad_ξ *descends to $H/H\xi$ and we have isomorphisms*

$$H^0(\mathbb{H}_k) \cong \mathbb{I}(H\xi)/H\xi \cong (H/H\xi)^{\operatorname{ad}_\xi} \tag{4.25}$$

Proof. First, it is clear that the action ad_ξ descends to the quotient. In particular, for $\alpha \in H/H\xi$, $[\xi, \alpha + H\xi] = [\xi, \alpha] + H\xi$. Next, we construct an injective map $\mathbb{I}(H\xi)/H\xi \hookrightarrow H/H\xi$. Consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & H\xi \arrow[r] \arrow[d, equal] & \mathbb{I}(H\xi) \arrow[r] \arrow[d, hook] & \mathbb{I}(H\xi)/H\xi \arrow[r] & 0 \\
0 \arrow[r] & H\xi \arrow[r] & H \arrow[r] & H/H\xi \arrow[r] & 0
\end{tikzcd}$$

The inclusion $\mathbb{I}(H\xi) \hookrightarrow H$ descends to a map $\mathbb{I}(H\xi)/H\xi \to H/H\xi$. Using the Snake lemma once again, yields an inclusion $\mathbb{I}(H\xi)/H\xi \hookrightarrow H/H\xi$. We show next that the image of $\mathbb{I}(H\xi)/H\xi$ lands inside $(H/H\xi)^{\mathrm{ad}_\xi}$. Indeed, if $\alpha + H\xi \in \mathbb{I}(H\xi)/H\xi$, then

$$[\xi, \alpha + H\xi] = [\xi, \alpha] + H\xi = \xi\alpha + H\xi$$

But $\alpha \in \mathbb{I}(H\xi)$, and thus there exists $\beta \in H$ such that $\xi\alpha = \beta\xi$. In conclusion, $[\xi, \alpha + H\xi] \in H\xi$ and ad_ξ acts trivially on the image of $\mathbb{I}(H\xi)/H\xi$ inside $H/H\xi$. In other words

$$\mathbb{I}(H\xi)/H\xi \hookrightarrow (H/H\xi)^{\mathrm{ad}_\xi} \tag{4.26}$$

To show that (4.26) is an isomorphism, let $\alpha + H\xi \in (H/H\xi)^{\mathrm{ad}_\xi}$, then notice that $[\xi, \alpha] \in H\xi$ implies that $\xi\alpha \in H\xi$ and thus $\alpha + H\xi \in \mathbb{I}(H\xi)/H\xi$. $\square$

Proof of Theorem for 4.5.3. The cohomology in degree 1 and -1 have already been taken care of. Now, recall that for $\tau \in \mathrm{Irr}(W)$, we have an isomorphism

$$H \otimes_{\mathbb{C}[V^*] \rtimes W} \tau \cong (H/H\xi) \otimes_{\mathbb{C}W} \tau$$

In particular, we can take $\tau = \mathbb{C}W$ and we see that

$$\begin{aligned} H/H\xi &\cong H/H\xi \otimes_{\mathbb{C}W} \mathbb{C}W \\ &\cong H/H\xi \otimes_{\mathbb{C}W} \left(\bigoplus_{\tau\in \mathrm{Irr}(W)} \tau\otimes\tau^* \right) \\ &= \bigoplus_{\tau\in \mathrm{Irr}(W)} (H/H\xi \otimes_{\mathbb{C}W} \tau)\otimes\tau^* \\ &= \bigoplus_{\tau\in \mathrm{Irr}(W)} M(\tau)\otimes\tau^* \end{aligned}$$

In particular,

$$H^0(\mathbb{H}_k) \cong (H/H\xi)^{\mathrm{ad}_\xi} \cong \bigoplus_{\tau\in \mathrm{Irr}(W)} M(\tau)^\xi \otimes\tau^*$$

and this concludes the proof of the theorem. □

4.6 Spherical subalgebra

Recall the definition of $\mathbf{e} := \frac{1}{|W|}\sum_{w\in W} w \in \mathbb{C}W \hookrightarrow \mathcal{D}(\Omega V_{reg}) \rtimes W$ and give the following definition:

Definition 4.6.1. The spherical subalgebra is the algebra $\boldsymbol{e}\mathbb{H}_k\boldsymbol{e} \subset \mathcal{D}(\Omega V_{reg}) \rtimes W$.

We need to explain this definition. Viewing $\mathbb{H}_k \hookrightarrow \mathcal{D}(\Omega V_{reg}) \rtimes W$, we can perform the multiplication $\boldsymbol{e}\mathbb{H}_k\boldsymbol{e}$ inside $\mathcal{D}(\Omega V_{reg}) \rtimes W$. It is not clear at first that the spherical subalgebra sits inside (the image of) $\mathbb{H}_k$, but in fact it does. Indeed, recall that Proposition 3.6.4 showed that, acting on forms

$$s = \frac{1}{2}(s\otimes 1)\left(i_{\alpha_s^\vee} m_{d\alpha_s} - m_{d\alpha_s} i_{\alpha_s^\vee}\right) \tag{4.27}$$

where s is the diagonal action on ΩV, and $s\otimes 1$ acts on ΩV through the tensor product of the canonical representation of W on $\mathbb{C}[V]$ and the trivial one on $\bigwedge V^*$.

This obviously still holds when acting on ΩV_{reg}. Hence, recalling the definition of

$$\Delta_s = \frac{1}{2}(\alpha_s^\vee \alpha_s - \alpha_s \alpha_s^\vee)$$

we see that $s \otimes \Delta_s \in \mathbb{H}_k$ acts on forms as the diagonal action of W. Since W is generated by reflections, we see that $\boldsymbol{e}$ is in fact in the image of $\mathbb{H}_k$ and thus so is $\boldsymbol{e}\mathbb{H}_k\boldsymbol{e}$.

We will thus think of $\boldsymbol{e}\mathbb{H}_k\boldsymbol{e} \subset \mathbb{H}_k$, where $\boldsymbol{e}$ is really the preimage of $\frac{1}{|W|}\sum_{w\in W} w$ inside $\mathbb{H}_k$.

To conclude this discussion on group actions, we include the following result which is a direct consequence of Proposition 3.6.3

Proposition 4.6.2. *The following formula holds in* $\mathbb{H}_k$

$$(s \otimes \Delta_s)(\alpha \otimes \beta)(s \otimes \Delta_s) = {}^s\alpha \otimes {}^s\beta \tag{4.28}$$

for $\alpha \in H_k$, $\beta \in Cl(V \oplus V^*)$ *and* $s \in \Sigma$. *In other words, conjugation with* $s \otimes \Delta_s$ *inside* $\mathcal{H}_k$ *acts as the diagonal action of* $\mathbb{C}W$ *on* $H_k \otimes Cl(V \oplus V^*)$ *(recall that* $\Delta_s^2 = 1$*).*

Theorem 4.6.3. *The differential on* $\mathbb{H}_k$ *leaves* $\boldsymbol{e}\mathbb{H}_k\boldsymbol{e}$ *stable, and thus* $\boldsymbol{e}\mathbb{H}_k\boldsymbol{e}$ *is a differential graded algebra.*

Proof. We need to show that $d\boldsymbol{e} = 0$. First, for s a reflection in W, let us compute $d(1 \otimes \Delta_s)$. Recall that $\Delta_s = \frac{1}{2}\left((1 \otimes \alpha_s^\vee)(1 \otimes \alpha_s) - (1 \otimes \alpha_s)(1 \otimes \alpha_s^\vee)\right)$, and thus

$$\begin{aligned} d(1 \otimes \Delta_s) &= \frac{1}{2} d\left((1 \otimes \alpha_s^\vee)(1 \otimes \alpha_s) - (1 \otimes \alpha_s)(1 \otimes \alpha_s^\vee)\right) \\ &= \frac{1}{2} d(1 \otimes \alpha_s^\vee)(1 \otimes \alpha_s) + \frac{1}{2}(1 \otimes \alpha_s) d(1 \otimes \alpha_s^\vee) \\ &= \alpha_s^\vee \otimes \alpha_s \end{aligned}$$

because $d(1 \otimes \alpha_s) = 0$ and the sign change comes from $\deg(1 \otimes \alpha_s) = 1$.

Computing $d(s \otimes \Delta_s)$, we have

$$\begin{aligned} d(s \otimes \Delta_s) &= d(s \otimes 1)(1 \otimes \Delta_s) + (s \otimes 1)d(1 \otimes \Delta_s) \\ &= -s\alpha_s^\vee \otimes \frac{1}{2}\alpha_s\alpha_s^\vee\alpha_s + s\alpha_s^\vee \otimes \alpha_s \\ &= -s\alpha_s^\vee \otimes \frac{1}{2}\alpha_s(2 - \alpha_s\alpha_s^\vee) + s\alpha_s^\vee \otimes \alpha_s \\ &= 0 \end{aligned}$$

The result follows by linearity, the Leibniz rule and from the fact that reflections generate W. $\square$

The following two properties follow immediately.

Proposition 4.6.4. *The restriction of the Dunkl embedding to $\boldsymbol{e}\mathbb{H}_k\boldsymbol{e}$ yields*

$$\mathrm{Res} : \boldsymbol{e}\mathbb{H}_k\boldsymbol{e} \hookrightarrow \boldsymbol{e}\mathcal{D}(\Omega V_{reg}) \rtimes W\boldsymbol{e} \xrightarrow{\sim} \mathcal{D}(\Omega V_{reg})^W$$

Also, localizing at δ yields an isomorphism $\boldsymbol{e}\mathcal{H}_k[\delta^{-1}]\boldsymbol{e} \simeq \mathcal{D}(\Omega V_{reg})^W$

Note that the isomorphism $\boldsymbol{e}\mathcal{D}W\boldsymbol{e} \xrightarrow{\sim} \mathcal{D}(\Omega V_{reg})^W$ is the inverse of

$$\mathcal{D}(V_{reg})^W \xrightarrow{\sim} \boldsymbol{e}\mathcal{D}W\boldsymbol{e}, \quad D \mapsto \boldsymbol{e}D\boldsymbol{e} = \boldsymbol{e}D = D\boldsymbol{e}$$

In other words, for $\boldsymbol{e}L\boldsymbol{e} \in \boldsymbol{e}\mathbb{H}_k\boldsymbol{e}$, $\mathrm{Res}\, L \in \mathcal{D}(\Omega V_{reg})^W$ is the invariant differential operator on ΩV_{reg} such that

$$\boldsymbol{e}(\mathrm{Res}\, L) = (\mathrm{Res}\, L)\boldsymbol{e} = \boldsymbol{e}L\boldsymbol{e} \tag{4.29}$$

The relationship between H_k and $\boldsymbol{e}H_k\boldsymbol{e}$ in the classical setting transfers to this situation and the following theorem holds.

Theorem 4.6.5. *If k is regular, then $\mathbb{H}_k$ and $\boldsymbol{e}\mathbb{H}_k\boldsymbol{e}$ are simple algebras, Morita equivalent to each other.*

Proof. If k is regular, H_k is a simple algebra and $\mathbb{H}_k = H_k \otimes Cl(V \oplus V^*)$ is isomorphic to a matrix algebra $M_{2^n}(H_k)$ with coefficients in the simple ring H_k. Hence, $\mathbb{H}_k$ is also simple. The same Morita argument used in the classical case yields here again that $\mathbb{H}_k$ and $\boldsymbol{e}\mathbb{H}_k\boldsymbol{e}$ are Morita equivalent, and thus both are simple algebras. $\square$

In finishing the picture on the spherical algebra, we offer the following conjecture.

Conjecture 4.6.6. If k is regular, then $(\Omega V)^W$ and $(\Omega V^*)^W$ generate $\boldsymbol{e}\mathbb{H}_k\boldsymbol{e}$.

Consider the maps defined as follows: for $\omega \in (\Omega V)^W$, write $\omega = \sum f_I dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ for $f \in \mathrm{Sym}(V^*)$ and $x_i \in V^*$, then

$$\begin{aligned} (\Omega V)^W &\longrightarrow \boldsymbol{e}\mathcal{H}_k\boldsymbol{e} \\ \sum f_I dx_{i_1} \wedge \cdots \wedge dx_{i_k} &\longmapsto \sum (f_I \otimes x_{i_1} \cdots x_{i_k})\boldsymbol{e} \end{aligned}$$

Similarly, write $\varpi \in (\Omega V^*)^W = \sum \varphi_I d\xi_{i_1} \wedge \cdots \wedge d\xi_{i_k}$, for $\varphi \in \mathrm{Sym}(V)$, and $\xi_i \in V$

$$\begin{aligned} (\Omega V^*)^W &\longrightarrow \boldsymbol{e}\mathcal{H}_k\boldsymbol{e} \\ \sum \varphi_I d\xi_{i_1} \wedge \cdots \wedge d\xi_{i_k} &\longmapsto \sum (\varphi_I \otimes \xi_{i_1} \cdots \xi_{i_k})\boldsymbol{e} \end{aligned}$$

Indeed, this is well defined as for say $\omega \in (\Omega V)^W$, $\omega\boldsymbol{e} = \boldsymbol{e}\omega\boldsymbol{e} = \boldsymbol{e}\omega\boldsymbol{e}$.

We now investigate the the situation in the rank 1 case $W = \mathbb{Z}/2\mathbb{Z}$. Recall that we have $V = \mathbb{C}\xi$, $V^* = \mathbb{C}x$, $\langle x, \xi \rangle = 1$, $W = \{1, s\}$, $\alpha_s = x$, $\alpha_s^\vee = 2\xi$, where the action of W on ΩV is $s \cdot (f dx) := ({}^s f) d({}^s x)$, and under the Dunkl embedding

$$\begin{aligned} s \otimes 1 &\mapsto \frac{1}{2}(i_{\alpha_s^\vee} m_{d\alpha_s} - m_{d\alpha_s} i_{\alpha_s^\vee}) s \\ &= (i_\xi m_{dx} - m_{dx} i_\xi) s \\ &= s(i_\xi m_{dx} - m_{dx} i_\xi) \end{aligned}$$

We give an explicit description of the spherical algebra in this context.

Theorem 4.6.7. *Let* $\mathbf{e} = e_0$ *and* e_1 *be the orthogonal idempotents of* $\mathbb{C}\mathbb{Z}_2$*, in other words* $e_0 := \frac{1}{2}(1+s), \quad e_1 = \frac{1}{2}(1-s)$*. Then spherical subalgebra* $\mathbf{e}\mathcal{H}_k\mathbf{e}$ *is isomorphic to the algebra*

$$\begin{pmatrix} e_1 H e_1 & e_1 H e_0 \\ e_0 H e_1 & e_0 H e_0 \end{pmatrix}$$

Proof. The orthogonal idempotents here are $\mathbf{e} = e_0 := \frac{1}{2}(1+s), \quad e_1 = \frac{1}{2}(1-s)$,

Under the identification $H_k \otimes Cl(V \oplus V^*) \simeq M_2(H_k)$ provided by

$$i_\xi \leftrightarrow 1 \otimes \xi \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$m_{dx} \leftrightarrow 1 \otimes x \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

and following $\mathbf{e} = \frac{1}{2}(1+s) = \frac{1}{2}\left(1 + s \otimes (\xi x - x\xi)\right)$, we get that

$$\mathbf{e} \mapsto \begin{pmatrix} \frac{1}{2}(1-s) & 0 \\ 0 & \frac{1}{2}(1+s) \end{pmatrix} = \begin{pmatrix} e_1 & 0 \\ 0 & e_0 \end{pmatrix}$$

A simple matrix computation then shows that under that identification

$$\mathbf{e}\mathcal{H}_k\mathbf{e} \mapsto \begin{pmatrix} e_1 H e_1 & e_1 H e_0 \\ e_0 H e_1 & e_0 H e_0 \end{pmatrix}$$

□

Conjecture 4.6.8. If k is regular, then $(\Omega\mathbb{C})^{\mathbb{Z}_2}$ and $(\Omega\mathbb{C}^*)^{\mathbb{Z}_2}$ generate $\mathbf{e}\mathcal{H}_k(\mathbb{Z}_2)\mathbf{e}$.

The invariant differential forms are here:

$$\begin{aligned}(\Omega V)^W &= f_0(x^2) + f_1(x^2)d(x^2)\\ &= f_0(x^2) + f_1(x^2)2xdx\\ (\Omega V^*)^W &= \varphi_0(\xi^2) + \varphi_1(\xi^2)d(\xi^2)\\ &= \varphi_0(\xi^2) + \varphi_1(\xi^2)2\xi d\xi\end{aligned}$$

and so the images of $(\Omega V)^W$ and $(\Omega V^*)^W$ inside $\mathbf{e}\mathcal{H}_k\mathbf{e}$ become here on generators:

$$(f_0(x^2) + f_1(x^2)d(x^2))\mathbf{e} \mapsto \begin{pmatrix} e_1 f_0(x^2)e_1 & e_1 f_1(x^2)2xe_0 \\ 0 & e_0 f_0(x^2)e_0 \end{pmatrix}$$

$$(\varpi_0(\xi^2) + \varpi_1(\xi^2)d(\xi^2))\mathbf{e} \mapsto \begin{pmatrix} e_1\varphi_0(\xi^2)e_1 & 0 \\ e_0\varphi_1(\xi^2)2\xi e_1 & e_0\varphi_0(\xi^2)e_0 \end{pmatrix}$$

So in the image we have (amongst other things):

$$\begin{pmatrix} e_1 & 0 \\ 0 & e_0 \end{pmatrix}, \quad \begin{pmatrix} 0 & e_1 x e_0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ e_0\xi e_1 & 0 \end{pmatrix}$$
$$\begin{pmatrix} e_1 x^2 e_1 & 0 \\ 0 & e_0 x^2 e_0 \end{pmatrix}, \quad \begin{pmatrix} e_1\xi^2 e_1 & 0 \\ 0 & e_0\xi^2 e_0 \end{pmatrix}$$

We are left to show that these elements generate the entire $\boldsymbol{e}\mathbb{H}_k\boldsymbol{e}$.

4.7 Quasi-invariant differential forms

4.7.1 Invariant differential forms

For W be a finite reflection group acting on a finite dimensional vector space V, Chevalley's Theorem asserts that the algebra of invariants $\mathbb{C}[V]^W$ is a polynomial algebra, i.e. there exists basic invariants $f_1, \cdots f_n \in \mathbb{C}[V]^W$ s.t. $\mathbb{C}[V]^W = \mathbb{C}[f_1, \cdots, f_n]$. Passing to differential forms, we have the following classic theorem due to L. Solomon:

Theorem 4.7.1 (Solomon [Sol63])**.** *Under the diagonal action of W on ΩV*

$$(\Omega V)^W = \mathbb{C}[f_1, \cdots, f_n] \otimes \bigwedge(df_1, \cdots, df_n)$$

Borrowing from ideas in the proof in [Sol63] we extract the following fact that we will need to perform computations with quasi-invariant differential forms. Throughout the rest of this section, W will be a finite Coxeter group.

Lemma 4.7.2. *Let W be a finite Coxter group and V its finite dimensional representation. Write $\mathbb{C}[V]^W = \mathbb{C}[f_1, \ldots, f_n]$, where the f_i's are fundamental invariants of W. Recall that $\delta = \prod_s \alpha_s$, then any $\omega \in \Omega V$ can be written in the form*

$$\omega = \sum \frac{u_{i_1 \ldots i_k}}{\delta} df_{i_1} \cdots df_{i_k} \tag{4.30}$$

where $u_{i_1 \ldots i_k} \in \mathbb{C}[V]$.

Proof. Let $\mathcal{K} = \mathbb{C}(V)$ denote the field of rational functions on V. Choosing a basis of V, we will identify $\mathbb{C}[V]$ with the algebra of polynomials $\mathbb{C}[x_1, \ldots x_n]$, where $n := \dim V$.

Following an argument similar to the one in the main theorem in [Sol63], the algebraic independence of the f_i's shows that the $\binom{n}{p}$ elements $df_{i_1}\cdots df_{i_p} \in \mathcal{K}\bigotimes\bigwedge^p V^*$ for $i_1 < \cdots < i_p$, are linearly independent over $\mathcal{K}$. As a vector space over $\mathcal{K}$, the dimension of $\mathcal{K}\otimes\bigwedge^p V^*$ is exactly $\binom{n}{p}$, and thus those elements provide a basis for $\mathcal{K}\otimes\bigwedge^p V^*$ over $\mathcal{K}$.

Let now $\omega = \sum_{i_1<\cdots<i_p} w_{i_1\cdots i_p} dx_{i_1}\cdots dx_{i_p} \in \Omega V$ of degree p. We can thus write

$$\omega = \sum_{i_1<\cdots<i_p} \frac{u_{i_1\cdots i_p}}{v_{i_1\cdots i_p}} df_{i_1}\cdots df_{i_p} \tag{4.31}$$

where $u_{i_1\cdots i_p}$ and $v_{i_1\cdots i_p} \in \mathbb{C}[V]$. Choosing a particular set of indices $I = \{i_1, \cdots, i_p\}$, multiply both sides of (4.31) with df_j for $j \in \{1, \cdots, n\} \setminus I$ so that all the terms in the summation vanish but one. Since $\omega \in \Omega V$, this yields an equation of the form:

$$\frac{u_{i_1\cdots i_p}}{v_{i_1\cdots i_p}} df_1\cdots df_n = p_{i_1\cdots i_p} dx_1\cdots dx_n \tag{4.32}$$

where $p_{i_1\cdots i_p} \in \mathbb{C}[V]$.

Let now J be the Jacobian of W, in other words

$$J := \det\left(\frac{\partial f_i}{\partial x_j}\right)_{i,j=1\ldots n}$$

then it is well known that for a complex reflection group W, the Jacobian is proportional to $\prod_s {\alpha_s}^{n_s-1}$ ([ST54], [Ste60]), which is simply δ when W is a Coxeter group. Equation (4.32) then becomes:

$$\frac{u_{i_1\cdots i_p}}{v_{i_1\cdots i_p}} \delta dx_1\cdots dx_n = p_{i_1\cdots i_p} dx_1\cdots dx_n$$

In other words, $\frac{u_{i_1\cdots i_p}}{v_{i_1\cdots i_p}}\delta \in \mathbb{C}[V]$, and the claim is proved. $\square$

Define here again $\mathcal{A}$ to be the set of all reflection hyperplanes in V, fixed by the generating reflections in W. For each reflection hyperplane $H \in \mathcal{A}$, let $\alpha_H \in V^*$

such that $H = \ker \alpha_H$, and finally, let $k : \mathcal{A}/W \to \mathbb{N}$ be an integer valued function on the hyperplanes, constant on conjugacy classes.

4.7.2 Quasi-invariant differential forms

Recall that in the definition of standard quasi-invariants, one can replace the condition

$$(1-s)f \equiv 0 \mod \langle \alpha_s \rangle^{2k_s}$$

for $f \in \mathbb{C}[V]$, with the equivalent one

$$(1-s)f \equiv 0 \mod \langle \alpha_s \rangle^{2k_s+1}$$

Indeed, working with $\mathbb{Z}/2$ acting on $\mathbb{C}[x]$ as in Example 2.4.12, the condition $(1-s)f \equiv 0 \mod \langle x \rangle^{2k}$ means that all the monomials in f of odd degree must have degree at least $2k$, or equivalently, at least $2k+1$, since they have odd degree.

When working with differential forms, on the other hand, those are *not equivalent* anymore. Indeed, $(1-s)(x^2 dx) = 2x^2 dx \equiv 0 \mod \langle x \rangle^2$, and yet, it is clearly not 0 modulo $\langle x \rangle^3$. Either condition has drawbacks and advantages when used to define quasi-invariant differential forms. We clarify this statement in this section.

Following the definition in the classical case, we propose (notice that k is integral here).

Definition 4.7.3. (for W a Coxeter group) The candidate sets of *k-quasi-*

invariants differential forms of W on V are

$$\mathcal{Q}_k(W) := \left\{\omega \in \Omega V : {}^{s_H}\omega \equiv \omega \mod \langle\alpha_H\rangle^{2k_H}, \text{ for all } H \in \mathcal{A}\right\} \tag{4.33}$$

$$\mathcal{Q}'_k(W) := \left\{\omega \in \Omega V : {}^{s_H}\omega \equiv \omega \mod \langle\alpha_H\rangle^{2k_H+1}, \text{ for all } H \in \mathcal{A}\right\} \tag{4.34}$$

where $\langle\alpha_H\rangle$ is the ideal generated by α_H inside of ΩV, and s_H is the unique element in W_H of order 2 and determinant -1.

Let us now mention yet another possible approach to defining quasi-invariant differential forms. We start by rewriting the definition of standard quasi-invariants in terms of decreasing filtrations of $A := \mathbb{C}[V]$. For $s \in \Sigma$, let I_s be the ideal $\langle{\alpha_s}^2\rangle \subset A$ and consider the filtration on A generated by the powers of I_s. More precisely, let $F_s^k A := I_s^k$ and write $\pi_{k,s} : A \twoheadrightarrow A/F_s^{k_s}A$ for the canonical projection. Then it is clear that we can rephrase the definition of standard quasi-invariants as follows:

$$f \in Q_k \Longleftrightarrow \pi_{k,s}(f) \in \left(A/F_s^{k_s}A\right)^{W_{H_s}} \ \ \forall s \in \Sigma$$

When passing from $\mathbb{C}[V]$ to ΩV, it is natural to consider replacing the I_s-adic filtration with a Hodge filtration, whose definition we now recall. Given an ideal $I \subset \mathbb{C}[V]$, the Hodge filtration on ΩV associated to I is the filtration defined by

$$F^k\Omega V := \bigoplus_{\substack{m+p=k \\ m,p\geq 0}} I^m\Omega^p V \quad k \geq 0$$

It is well known that when $\mathrm{Spec}(\mathbb{C}[V]/I)$ is a singular variety, the Kähler forms and the de Rham complex are not the right objects to look at. Indeed, the Kähler forms do not coincide with the regular ones, and the (hyper)cohomology of the de Rham complex does not generally coincide with the singular cohomology of $\mathrm{Spec}(\mathbb{C}[V]/I)$. In this case, the de Rham complex has to be replaced with the complexes $\Omega V/F^n\Omega V$. Let $B := \mathbb{C}[V]/I$ and denote by $H_n^\bullet(B)$ the cohomology of the complexes $\Omega V/F^n\Omega V$. Then it is well known that the $H^\bullet_n(B)$ do not depend on the embedding $\mathrm{Spec}(B) \hookrightarrow \mathrm{Spec}(\mathbb{C}[V])$, and there are isomorphisms (see [FT85]):

$$\varprojlim_n H^\bullet_n(B) \cong HH^\bullet(B)$$

In light of the previous discussion, one can consider replacing the I-adic filtrations in the definition of quasi-invariant differential forms with Hodge filtrations. We do not follow this direction here (other that in Example 4.7.6) and focus on $\mathcal{Q}_k$ and $\mathcal{Q}'_k$. We start with the following theorem:

Theorem 4.7.4. *Let W be a Coxeter group acting on V and let Q_k be its space of quasi-invariant functions. Then we have*

$$\mathcal{Q}'_k = Q_k \otimes_{\mathbb{C}[V]^W} (\Omega V)^W \subset \;\mathcal{Q}_k\; \subset Q_{k-1} \otimes_{\mathbb{C}[V]^W} (\Omega V)^W, \quad \textit{if } k_s \geq 1 \textit{ for all } s$$

In other words

$$\mathcal{Q}'_k \subset \mathcal{Q}_k \subset \mathcal{Q}'_{k-1} \tag{4.35}$$

where $\mathcal{Q}'_k = Q_k \otimes_{\mathbb{C}[V]^W} (\Omega V)^W$.

Remark 4.7.5. If $\mathbb{C}[V]^W = \mathbb{C}[f_1, \ldots, f_n]$ where $f_1, \ldots, f_n$ are fundamental invariants of W, then

$$\mathcal{Q}'_k = Q_k \otimes_{\mathbb{C}[V]^W} (\Omega V)^W \simeq Q_k \otimes \bigwedge(df_1, \ldots, df_n)$$

Before we prove this theorem, let us give an example.

Example 4.7.6. Let $\mathbb{Z}/2$ act on $V = \mathbb{C}\xi$ and $V^* = \mathbb{C}x$ as in Example 2.4.12. For $\omega = p + q dx \in \Omega V$

$$(1-s)\omega = (p - {}^s p) + (q + {}^s q)dx$$

(i) The condition $(1-s)\omega \equiv 0 \mod \langle x\rangle^{2k}$ is equivalent to

$$p - {}^s\!p \in x^{2k}\mathbb{C}[x], \qquad q + {}^s\!q \in x^{2k}\mathbb{C}[x]$$

and thus $p \in Q_k$ while

$$q \in x\left(x^{2(k-1)+1}\mathbb{C}[x^2] + \mathbb{C}[x^2]\right) = xQ_{k-1}$$

which shows that

$$\mathcal{Q}_k = Q_k + Q_{k-1}d(x^2)$$

(ii) The condition $(1-s)\omega \equiv 0 \mod \langle x\rangle^{2k+1}$ is equivalent to

$$p - {}^s\!p \in x^{2k+1}\mathbb{C}[x], \qquad q + {}^s\!q \in x^{2k+1}\mathbb{C}[x]$$

Here again, $p \in Q_k$, but since $q + {}^s\!q$ has all even degrees, $q + {}^s\!q \in x^{2k+2}\mathbb{C}[x]$ in fact and thus

$$q \in x\left(x^{2k+1}\mathbb{C}[x^2] + \mathbb{C}[x^2]\right) = xQ_k$$

and thus here

$$\mathcal{Q}'_k = Q_k + Q_k d(x^2)$$

(iii) Let us now consider the Hodge filtration on ΩV generated by $I = \langle x^2\rangle$. Here, we have

$$F^k\Omega V = x^{2k}\mathbb{C}[x] \oplus x^{2(k-1)}\mathbb{C}[x]dx$$

for $k \geq 1$. Thus $\omega = p + qdx \in \left(\Omega V / F^k\Omega V\right)^W$ is equivalent to the conditions

$$p - {}^s\!p \in x^{2k}\mathbb{C}[x], \qquad q + {}^s\!q \in x^{2(k-1)}\mathbb{C}[x]$$

hence, $p \in Q_k$, while $q \in xQ_{k-2}$.

This example shows that one should expect the inclusions in Theorem 4.7.4 to be strict.

Proof of Theorem 4.7.4. (i) First, let us show that $\mathcal{Q}_k \subset Q_{k-1} \otimes \bigwedge(df_1, \dots, df_n)$. Let $\omega \in \mathcal{Q}_k$, and use Lemma 4.7.2 to write

$$\omega = \sum_{i_1 < \cdots < i_p} \frac{u_{i_1 \dots i_p}}{\delta} df_{i_1} \cdots df_{i_p}$$

where $u_{i_1 \dots i_p} \in \mathbb{C}[V]$. For a reflection $s \in W$ we have:

$$(1-s)\omega = \sum_{i_1 < \cdots < i_p} \left(\frac{u_{i_1 \dots i_p}}{\delta} + \frac{{}^s u_{i_1 \dots i_p}}{\delta} \right) df_{i_1} \cdots df_{i_p}$$

Since $\omega \in \mathcal{Q}_k$, we can write $(1-s)\omega = {\alpha_s}^{2k_s} \omega'$, where $\omega' \in \Omega V$. Multiplying by appropriate subsets of f_i's as in the proof of lemma 4.7.2, yields the following equations for all $i_1 < \cdots < i_p$:

$$\left(\frac{u_{i_1 \dots i_p} + {}^s u_{i_1 \dots i_p}}{\delta} \right) \delta dx_1 \cdots dx_n = {\alpha_s}^{2k_s} p_{i_1 \dots i_p} dx_1 \cdots dx_n \tag{4.36}$$

where again $p_{i_1 \dots i_p} \in \mathbb{C}[V]$. Therefore

$$u_{i_1 \dots i_p} + {}^s u_{i_1 \dots i_p} \in {\alpha_s}^{2k_s} \mathbb{C}[V] \tag{4.37}$$

Diagonalizing s and thus essentially working in $\mathbb{C}[x]$, one immediately concludes in particular that if $k \geq 1$, $u_{i_1 \cdots i_p} \in \alpha_s \mathbb{C}[V]$. Repeating this procedure for all reflections s shows that $u_{i_1 \dots i_p} \in \delta \mathbb{C}[V]$, and thus we can write

$$\omega = \sum_{i_1 < \cdots < i_p} v_{i_1 \dots i_p} df_{i_1} \cdots df_{i_p}$$

in which case

$$(1-s)\omega = \sum_{i_1 < \cdots < i_p} \left(v_{i_1 \cdots i_p} - {}^s v_{i_1 \cdots i_p} \right) df_{i_1} \cdots df_{i_p}$$

Since $(1-s)w = {\alpha_s}^{2k_s} \omega'$ for some $\omega' \in \Omega V$, repeat the trick that lead to equations (4.36) and (4.37) to obtain

$$\left(v_{i_1 \cdots i_p} - {}^s v_{i_1 \dots i_p} \right) \delta \in {\alpha_s}^{2k_s} \mathbb{C}[V] \tag{4.38}$$

It follows that $v_{i_1 \cdots i_p} - {}^s v_{i_1 \dots i_p} \in {\alpha_s}^{2(k_s - 1)+1} \mathbb{C}[V]$ for all s, and thus $v_{i_1 \cdots i_p} \in Q_{k-1}$.

(ii) To show that $\mathcal{Q}'_k \subset Q_k \otimes \bigwedge(df_1, \ldots, df_n)$, we repeat the procedure above and find this time that if $\omega = \sum_{i_1 < \cdots < i_p} v_{i_1 \ldots i_p} df_{i_1} \cdots df_{i_p} \in \mathcal{Q}'_k$, then

$$\left(v_{i_1 \cdots i_p} - {}^s v_{i_1 \ldots i_p}\right) \delta \in {\alpha_s}^{2k_s+1} \mathbb{C}[V] \tag{4.39}$$

It follows that $v_{i_1 \cdots i_p} - {}^s v_{i_1 \ldots i_p} \in {\alpha_s}^{2k_s} \mathbb{C}[V]$ for all s, and thus $v_{i_1 \cdots i_p} \in Q_k$.

(iii) Finally, we show that $Q_k \otimes \bigwedge(df_1, \ldots, df_n)$ sits inside both $\mathcal{Q}_k$ and $\mathcal{Q}'_k$. Let $\omega \in Q_k \otimes \bigwedge(df_1, \ldots, df_n)$. Then we can write

$$\omega = \sum_{i_1 < \cdots < i_p} v_{i_1 \ldots i_p} df_{i_1} \cdots df_{i_p}$$

where $v_{i_1 \ldots i_p} \in Q_k$. For a reflection s we have once again

$$(1-s)\omega = \sum_{i_1 < \cdots < i_p} \left(v_{i_1 \cdots i_p} - {}^s v_{i_1 \cdots i_p}\right) df_{i_1} \cdots df_{i_p}$$

Since $v_{i_1 \cdots i_p} \in Q_k$, $v_{i_1 \cdots i_p} - {}^s v_{i_1 \cdots i_p} \in {\alpha_s}^{2k_s+1} \mathbb{C}[V]$, and the conclusions follow.

$\square$

An immediate consequence is then the following theorem:

Theorem 4.7.7. *Let W a finite Coxeter group. Then the following properties hold:*

(i) $\mathcal{Q}_0 := \Omega V$ *and* $\mathcal{Q}_\infty := \bigcap_k \mathcal{Q}_k = (\Omega V)^W$ *(and similarly for $\mathcal{Q}'_k$).*

(ii) $(\Omega V)^W \subset \mathcal{Q}_k \subset (\Omega V)$ *for all k, and $\mathcal{Q}_k \subset \mathcal{Q}_{k'}$ whenever $k \geq k'$ (and similarly for $\mathcal{Q}'_k$).*

(iii) $\mathcal{Q}_k$ *and* $\mathcal{Q}'_k$ *are finite $(\Omega V)^W$-modules and finitely generated algebras.*

(iv) ΩV *is a finite $\mathcal{Q}_k$-module (resp. $\mathcal{Q}'_k$).*

Proof. The proofs are clear given Theorem 2.1.18 and the structure of $\mathcal{Q}'_k$ given by theorem 4.7.4. The only non obvious statements are (iii) and (iv). (iii) follows from the fact that Q_k is a finitely generated $\mathbb{C}[V]^W$-module, and so $\mathcal{Q}'_k$ is a finitely generated $(\Omega V)^W$-module. Since $(\Omega V)^W$ is left Noetherian, $\mathcal{Q}_k \subset \mathcal{Q}_{k-1}$ is also finitely generated. For (iv) $\mathbb{C}[V]$ is a finitely generated $\mathbb{C}[V]^W$-module, and thus so is $\Omega V = \mathbb{C}[V] \otimes \bigwedge V^*$ since $\bigwedge V^*$ is in fact finite dimensional over $\mathbb{C}$. A fortiori, ΩV is finitely generated over $\mathcal{Q}_k$ and $\mathcal{Q}'_k$, as both contain $\mathbb{C}[V]^W$. $\square$

The following is also immediate.

Theorem 4.7.8. *$\mathcal{Q}'_k$ is a free $(\Omega V)^W$-module for all k.*

Proof. We know from theorem 2.1.20 that Q_k is a free $\mathbb{C}[V]^W$-module, i.e. $Q_k \simeq (\mathbb{C}[V]^W)^{\oplus I}$. But then $\mathcal{Q}'_k = Q_k \otimes_{\mathbb{C}[V]^W} (\Omega V)^W \simeq (\mathbb{C}[V]^W)^{\oplus I} \otimes_{\mathbb{C}[V]^W} (\Omega V)^W \simeq ((\Omega V)^W)^{\oplus I}$, and $\mathcal{Q}_k$ is a free $(\Omega V)^W$-module. $\square$

While Example 4.7.6 shows that $\mathcal{Q}_k(\mathbb{Z}/2)$ is free over $(\Omega V)^W$, it is not obvious at this point whether this property holds in general. Theorems 4.7.7 and 4.7.8 seem to indicate that $\mathcal{Q}'_k$ might be the right notion of quasi-invariant differential forms. However, together with the description of the structure of $\mathcal{Q}'_k$ given in Theorem 4.7.4, they also suggest that this notion of quasi-invariant differential forms doesn't provide anything new. In the next section, we investigate the relationship between quasi-invariant differential forms and the graded Cherednik algebra.

4.7.3 $\mathbb{C}W$-valued quasi-invariants

Following [BC11], we view $\mathcal{D}W := \mathcal{D}(\Omega V_{reg}) \rtimes W$ as a ring of W-equivariant differential operators on ΩV_{reg}. As such it acts naturally on the space $\Omega V_{reg} \otimes \mathbb{C}W$ of $\mathbb{C}W$-valued differential forms. More precisely, using the canonical inclusion $\Omega V_{reg} \otimes \mathbb{C}W \to \mathcal{D}W$, we can identify $\Omega V_{reg} \otimes \mathbb{C}W$ with the $\mathcal{D}W$-module $\mathcal{D}W/\mathcal{D}W \langle \mathcal{L}_\xi, i_\xi, \xi \in V\rangle$. Explicitly, in terms of generators, $\mathcal{D}W$ acts on $\Omega V_{reg} \otimes \mathbb{C}W$ by:

$$\begin{aligned}
\omega(\phi \otimes u) &= \omega\phi \otimes u, \quad \omega \in \Omega V_{reg} \\
\mathcal{L}_\xi(\phi \otimes u) &= (\mathcal{L}_\xi \phi) \otimes u, \quad \xi \in V \\
i_\xi(\phi \otimes u) &= (i_\xi \phi) \otimes u, \quad \xi \in V \\
s(\phi \otimes u) &= {}^s\phi \otimes su, \quad s \in W
\end{aligned}$$

This is because, for example, $i_\xi m_\phi = i_\xi m_\phi - (-1)^{|\phi|} m_\phi i_\xi$ modulo $\mathcal{D}W \langle \mathcal{L}_\xi, i_\xi, \xi \in V\rangle$, which is easily seen to act as the multiplication by $i_\xi \phi$. Recall now that we have the Dunkl embedding $\mathbb{H}_k \to \mathcal{D}W$, given on generators by:

$$\begin{aligned}
& s \otimes 1 \mapsto \frac{1}{2}\left(i_{\alpha_s^\vee} m_{d\alpha_s} - m_{d\alpha_s} i_{\alpha_s^\vee}\right) s \qquad (4.40)\\
& \xi \otimes 1 \mapsto \tilde{\mathcal{L}}_\xi, \qquad x \otimes 1 \mapsto m_x \\
& 1 \otimes \xi \mapsto i_\xi, \qquad 1 \otimes x \mapsto m_{dx}
\end{aligned}$$

for $\xi \in V$, $x \in V^*$. Restriction of scalars along the Dunkl embedding thus makes $\Omega V_{reg} \otimes \mathbb{C}W$ an $\mathbb{H}_k$-module.

Besides this diagonal action of W onto $\Omega V_{reg} \otimes \mathbb{C}W$, we also consider the action which is trivial on the first factor. More precisely, for $w \in W$ we denote this action by $1 \otimes w$ and by definition $w(\phi \otimes u) := \phi \otimes wu$, where $\phi \otimes u \in \Omega V_{reg} \otimes \mathbb{C}W$.

Theorem 4.7.9 (Differential version of [BC11] Theorem 3.4)**.** *Let W be a finite Coxeter group. If k is integral, then $\Omega V_{reg} \otimes \mathbb{C}W$ contains a unique dg $\mathbb{H}_k$-submodule $\mathbf{Q}_k = \mathbf{Q}_k(W)$, such that $\mathbf{Q}_k$ is finite over ΩV and*

$$\mathbf{e}\mathbf{Q}_k = \mathbf{e}(\mathcal{Q}_k \otimes 1) \tag{4.41}$$

where $\mathcal{Q}_k \subset \Omega V$ is the space of quasi-invariant differential forms of W as defined in (4.33).

At this point, one might wonder why we chose to single out $\mathcal{Q}_k$. It turns out, as the proof of Theorem 4.7.9 will show, that choosing $\mathcal{Q}'_k$ does *not* yield a module structure on the graded Cherednik algebra.

To prove this theorem, we show that the subspace

$$\mathbf{Q}_k(W) := \left\{\varphi \in \Omega V \otimes \mathbb{C}W \ : \ (1 \otimes e_{s,i})\varphi \equiv 0 \mod \langle \alpha_s \rangle^{2k_s} \otimes \mathbb{C}W, s \in \Sigma, i = 0, 1\right\} \tag{4.42}$$

satisfies the required properties. Here $\langle \alpha_s \rangle$ stands for the ideal generated by α_s inside ΩV. The following proofs are adapted to the differential case from the ones in the classical case, given in [BC11].

Lemma 4.7.10. *The subspace $\mathbf{Q}_k$ as defined by (4.42) satisfies (4.41)*

Proof. The argument in the classical cases transports easily here. Indeed, we need to prove that $\boldsymbol{e}(\omega \otimes 1) \in \boldsymbol{e}\mathbf{Q}_k$ if and only if $\omega \in \mathcal{Q}_k$. For any given $\omega \in \Omega V$ and $s \in W$, compute

$$(1 \otimes s)[\boldsymbol{e}(\omega \otimes 1)] = \frac{1}{|W|}\sum_{w \in W} {}^{w}\omega \otimes sw = \frac{1}{|W|}\sum_{w \in W} {}^{s^{-1}w}\omega \otimes w$$

Multiplying by the appropriate characters and summing up over all $s \in W_J$, we get

$$(1 \otimes e_{s,i})[\boldsymbol{e}(\omega \otimes 1)] = \frac{1}{|W|}\sum_{w \in W} e_{s,-i}({}^{w}\omega) \otimes w$$

Thus by definition, $\boldsymbol{e}(\omega \otimes 1) \in \boldsymbol{e}\mathbf{Q}_k$ if and only if ${}^w\omega \in \mathcal{Q}_k$ for all $w \in W$. The latter is equivalent to $\omega \in \mathcal{Q}_k$, since it is clear that $\mathcal{Q}_k$ is W-stable. $\square$

Lemma 4.7.11. *The space $\mathbf{Q}_k$ is stable under the action of $\mathbb{H}_k$ on $\mathbf{Q}_k$ through the Dunkl embedding $\mathbb{H}_k \hookrightarrow \mathcal{D}W$.*

Proof. It is clear from the definition of $\mathbf{Q}_k$ and the expression of the Dunkl embedding given in (4.40) that $\mathbf{Q}_k$ is stable under the action of the generators $x \otimes 1$, $1 \otimes x$ and $1 \otimes x$ of $\mathbb{H}_k$, for $x \in V^*$ and $\xi \in V$. We then show that $\mathbf{Q}_k$ is stable under the diagonal action of $\mathbb{C}W \subset \mathcal{D}W$. This, along with the identity

$$s \otimes 1 \mapsto \frac{1}{2}\left(i_{\alpha_s^\vee} m_{d\alpha_s} - m_{d\alpha_s} i_{\alpha_s^\vee}\right) s$$

will show that $\mathbf{Q}_k$ is stable under the action of $s \otimes 1$, for reflections $s \in W$, which generate W.

To see that $\mathbf{Q}_k$ is stable under the diagonal action of $\mathbb{C}W$, observe then that for $w \in W$ and H a reflection hyperplane

$$\begin{aligned} w(1 \otimes e_{H,i}) &= w \otimes \left(\frac{1}{|W|}\sum_{s \in W_H} \det(s)^{-1} ws\right) \\ &= w \otimes \left(\frac{1}{|W|}\sum_{s \in W_H} \det(wsw^{-1})^{-1} wsw^{-1}\ w\right) \\ &= (1 \otimes e_{wH})w \end{aligned}$$

as endomorphisms of $\Omega V_{reg} \otimes \mathbb{C}W$. Since (4.42) holds for all reflection hyperplanes and the k's only depend on the conjugation class of the hyperplanes, we have that $w\mathbf{Q}_k \subset \mathbf{Q}_k$ for all $w \in W$, and thus $(w \otimes 1)\mathbf{Q}_k \subset \mathbf{Q}_k$ as well.

Just as in the classical case, the only non-trivial fact is that the differential action of $\xi \otimes 1$ on $\Omega V_{reg} \otimes \mathbb{C}W$ preserves $\mathbf{Q}_k$. Just as in the classical case [BC11, Lemma 3.6], the statement can be reduced to be checked in dimension 1, and we provide the proof in that case.

Let $W = \mathbb{Z}/2$ act on $V = \mathbb{C}\xi$ and $V^* = \mathbb{C}x$ as in Example 2.4.12. We have here $e_0 = \frac{1}{2}(1+s)$ and $e_1 = \frac{1}{2}(1-s)$. Clearly $\mathbf{Q}_k = \Omega V \otimes e_0 + x^{2k}\Omega V \otimes e_1$. Indeed, if $\varphi \in \mathbf{Q}_k$, write $\varphi = \omega_0 \otimes e_0 + \omega_1 \otimes e_1$, then:

$$\begin{aligned} 1 \otimes \frac{1}{2}(1-s)\varphi &= 1 \otimes \frac{1}{2}(1-s)\left(\omega_0 \otimes \frac{1}{2}(1+s) + \omega_1 \otimes \frac{1}{2}(1-s)\right) \\ &= \omega_1 \otimes \frac{1}{2}(1-s) \end{aligned}$$

The condition $\omega_1 \otimes \frac{1}{2}(1-s) \equiv 0 \mod \langle x\rangle^{2k} \otimes \mathbb{C}W$ then implies that $\omega_1 \in x^{2k}\Omega V$.

Now recall that under the Dunkl embedding

$$\xi \otimes 1 \mapsto \tilde{\mathcal{L}}_\xi = \mathcal{L}_\xi - \sum_s k_s \alpha_s^{-1}\langle \alpha_s, \xi\rangle(1 - (s \otimes 1))$$

where $(s \otimes 1)$ denotes the action of $\mathbb{C}W$ on ΩV_{reg} which is trivial on $\bigwedge V^*$. In other words,

$$\xi \otimes 1 \mapsto \tilde{\mathcal{L}}_\xi = \mathcal{L}_\xi - \sum_s k_s \alpha_s^{-1}\langle \alpha_s, \xi\rangle(1 - s\Delta_s)$$

where we abuse our previous notation slightly and write $\Delta_s = \frac{1}{2}(i_{\alpha_s^\vee} m_{d\alpha_s} - m_{d\alpha_s} i_{\alpha_s^\vee})$(see Proposition 3.6.4).

We only need to check that the action of $\xi \otimes 1 \in \mathbb{H}_k$ stabilizes $x^{2k}\Omega V \otimes e_1$. First, note that acting on $\mathbb{C}W$-valued forms, we have

$$(s\Delta_s) \cdot (\phi \otimes w) = ((1 \otimes s) \cdot \phi) \otimes sw$$

Finally, compute

$$\begin{aligned} \tilde{\mathcal{L}}_\xi(x^{2k}\omega \otimes e_1) &= \mathcal{L}_\xi(x^{2k})\omega \otimes e_1 + x^{2k}\mathcal{L}_\xi\omega \otimes e_1 - \frac{k}{x}(1 - s\Delta_s)(x^{2k}\omega \otimes e_1) \\ &= 2kx^{2k-1}\omega \otimes e_1 + x^{2k}\mathcal{L}_\xi\omega \otimes e_1 - kx^{2k-1}\left[\omega + (s \otimes 1) \cdot \omega\right] \otimes e_1 \\ &= 2kx^{2k-1}\frac{1}{2}\left[\omega - (s \otimes 1) \cdot \omega\right] \otimes e_1 + x^{2k}\mathcal{L}_\xi\omega \otimes e_1 \end{aligned}$$

But $[\omega - (s \otimes 1) \cdot \omega] \in x\Omega V$, and this concludes the proof. Note that this would *not* work if the action here were the diagonal one. $\square$

Remark 4.7.12. The previous theorem doesn't hold if one changes the condition in the definition of $\mathbb{C}W$-valued quasi-invariants to

$$(1 \otimes e_{s,i})\varphi \equiv 0 \mod \langle \alpha_s \rangle^{2k_s+1}$$

Indeed, in the proof that the Dunkl operators preserve $\mathbf{Q}_k$, it was crucial that the exponent of x was $2k$ and not $2k+1$. In the standard case, this was not a problem, as the definition of (non $\mathbb{C}W$-valued) quasi-invariant functions was insensitive to that change. In our situation though, the two definitions are not equivalent and one has to be careful.

Next, we investigate the differential structure on $\mathbf{Q}_k$. Note that the deformed de Rham differential $\tilde{d}$ is in fact an operator in $\mathcal{D}W$. Indeed, recall that taking orthogonal dual bases $\{\xi_i\}$ and $\{x_i\}$ of V and V^*, Proposition (4.3.9) yielded the following expression for $\tilde{d}$

$$\tilde{d} = \sum_i m_{dxi} \tilde{\mathcal{L}}_\xi \in \mathcal{D}W \tag{4.43}$$

As such, it acts through the differential action on $\mathbf{Q}_k \subset \Omega V_{reg} \otimes \mathbb{C}W$. We show that this action in fact preserves $\mathbf{Q}_k$.

Lemma 4.7.13. *The deformed de Rham differential $\tilde{d}$ is a differential on $\mathbf{Q}_k$. In other words*

$$\tilde{d} \cdot \mathbf{Q}_k \subset \mathbf{Q}_k$$

and $\mathbf{Q}_k$ equipped with $\tilde{d}$ and the grading coming from ΩV is a dg-module over $\mathbb{H}_k$.

Proof. The equality (4.43) shows that $\tilde{d}$ is in fact in the image of $\mathbb{H}_k$ inside $\mathcal{D}W$, which we just proved preserves $\mathbf{Q}_k$.

For the second part of the claim, recall that the differential on $\mathbb{H}_k$ was given by commutating with $\tilde{d}$. Hence for $D \in \mathbb{H}_k$ and $\theta \in \mathbf{Q}_k$, we have:

$$\begin{aligned}\tilde{d} \cdot (D \cdot \theta) &= (\tilde{d}\ D) \cdot \theta \\ &= (\tilde{d}D - (-1)^{|D|} D\tilde{d}) \cdot \theta + (-1)^{|D|} D\tilde{d} \cdot \theta \\ &= d(D) \cdot \theta + (-1)^{|\zeta|} D \cdot (\tilde{d} \cdot \theta)\end{aligned}$$

□

Lemma 4.7.14. *If k is integral, there exists at most one $\mathcal{H}_k$-submodule $\mathbf{Q}_k \subset \Omega V_{reg} \otimes \mathbb{C}W$, satisfying (4.41).*

Proof. The argument in [BC11] goes through unchanged. We include it for the sake of completeness. Suppose that $\mathbf{Q}_k$ and $\mathbf{Q}'_k$ are two such modules. Replacing one of them by their sum, we my assume that $\mathbf{Q}_k \subset \mathbf{Q}'_k$, $\boldsymbol{e}\mathbf{Q}_k = \boldsymbol{e}\mathbf{Q}'_k$. Setting $M := \mathbf{Q}'_k / \mathbf{Q}_k$, we get that $\boldsymbol{e}M = 0$. This forces $M = 0$, since by Morita equivalence the functor

$$\mathrm{Mod}(\mathbb{H}_k) \to \mathrm{Mod}(\boldsymbol{e}\mathbb{H}_k\boldsymbol{e}), \quad M \mapsto \boldsymbol{e}M$$

is an equivalence of categories, and hence fully faithful. Thus $\mathbf{Q}_k = \mathbf{Q}'_k$ as required.

□

Recall now that $\mathcal{Q}_k \subset \Omega V$, and as such, the deformed de Rham differential acts on it $\tilde{d}$. Obviously

$$\tilde{d} \cdot (\mathcal{Q}_k \otimes 1) = (\tilde{d}\mathcal{Q}_k) \otimes 1$$

where the dot notation was used to emphasize that the action on the left is that of $\tilde{d} \in \mathcal{D}W$ acting on $\Omega V_{reg} \otimes \mathbb{C}W$. We can now conclude this section with the following corollary.

Corollary 4.7.15. *$\mathcal{Q}_k$ is naturally a dg-module over the spherical sub-algebra $\boldsymbol{e}\mathcal{H}_k\boldsymbol{e}$, which acts on $\mathcal{Q}_k$ by invariant differential operators through the spherical Dunkl embedding* $\mathrm{Res} : \boldsymbol{e}\mathbb{H}_k\boldsymbol{e} \hookrightarrow \mathcal{D}(\Omega V_{reg})^W$.

Proof. First, notice that for $\omega \in \Omega V_{reg}$, the equation $\boldsymbol{e}(\omega \otimes 1) = 0$ in $\Omega V_{reg} \otimes \mathbb{C}W$ forces $\omega = 0$ (recall that W acts diagonally).

Let us show that $\tilde{d}\mathcal{Q}_k \subset \mathcal{Q}_k$. We have $\boldsymbol{e}\left[(\tilde{d}\mathcal{Q}_k) \otimes 1\right] = \boldsymbol{e}\tilde{d}(\mathcal{Q}_k \otimes 1)$. Now recall that we proved in Theorem 4.6.3 that $d\boldsymbol{e} = 0$ where $d = [\tilde{d}, -]$ is the differential on $\mathbb{H}_k$. In other words $[\tilde{d}, \boldsymbol{e}] = 0$ when acting of $\mathbb{C}W$-valued forms. We have $\boldsymbol{e}\left[(\tilde{d}\mathcal{Q}_k) \otimes 1\right] = \tilde{d}\boldsymbol{e}(\mathcal{Q}_k \otimes 1) = \tilde{d}\boldsymbol{e}\mathbf{Q}_k = \boldsymbol{e}\tilde{d}\mathbf{Q}_k \subset \boldsymbol{e}\mathbf{Q}_k$. Thus $\boldsymbol{e}\left[(\tilde{d}\mathcal{Q}_k) \otimes 1\right] \subset \boldsymbol{e}(\mathcal{Q}_k \otimes 1)$, and $\tilde{d}\mathcal{Q}_k \subset \mathcal{Q}_k$.

Next we show that $\mathcal{Q}_k$ is stable under the action of $\boldsymbol{e}\mathbb{H}_k\boldsymbol{e}$. For $\boldsymbol{e}L\boldsymbol{e} \in \boldsymbol{e}\mathbb{H}_k\boldsymbol{e}$, we have here again that $\boldsymbol{e}\left[\mathrm{Res}\, L(\mathcal{Q}_k) \otimes 1\right] = (\boldsymbol{e}\, \mathrm{Res}\, L)(\mathcal{Q}_k \otimes 1) = (\boldsymbol{e}L\boldsymbol{e})\boldsymbol{e}(\mathcal{Q}_k \otimes 1)$, where we used the equation given in (4.29). But it is clear that $\boldsymbol{e}L\boldsymbol{e}\mathbf{Q}_k \subset \mathbf{Q}_k$, and thus $\boldsymbol{e}\left[\mathrm{Res}\, L(\mathcal{Q}_k) \otimes 1\right] \subset \boldsymbol{e}(\mathcal{Q}_k \otimes 1)$. The conclusion follows. $\square$

APPENDIX A

COMPUTATIONS

In this section, we provide the computations for the relations in $\mathcal{H}_k$. Here again, W is a finite Coxeter group with finite dimensional reflection representation V. We let $A = \mathbb{C}[V]$ and we use the notations defined in section 4.2. For the sake of readability we will also use the notation $\hat{s} := s \otimes 1$ and $s := s \otimes s$.

Recall also that for $\xi \in V \subset \mathrm{Der}(A)$, we defined the operator

$$\tilde{\mathcal{L}}_\xi := [\tilde{d}, i_\xi] = \mathcal{L}_\xi - [\Omega, i_\xi] =: \mathcal{L}_\xi - \pi_\xi$$

Let us start with finding a better expression for π_ξ.

Lemma A.0.16. *For* $\xi \in V$

$$\pi_\xi = \sum_s k_s \alpha_s^{-1} \langle \alpha_s, \xi \rangle (1 - \hat{s})$$

Proof. For $f \in A$, we have:

$$\begin{aligned}
\pi_\xi f &= [\Omega, i_\xi] f \\
&= \Omega i_\xi f + i_\xi \Omega f \\
&= i_\xi (\sum_s k_s (1 - s)(f) \alpha_s^{-1} d\alpha_s) \\
&= \sum_s k_s (1 - s)(f) \alpha_s^{-1} i_\xi d\alpha_s \\
&= \sum_s k_s (1 - s)(f) \alpha_s^{-1} \langle \alpha_s, \xi \rangle
\end{aligned}$$

Doing it now for a general k-form $fdz_1\wedge\cdots\wedge dz_k$, where the z_i's $\in V^*$, we get:

$$\begin{aligned}
\pi_\xi(fdz_1\wedge\cdots\wedge dz_k) &= [\Omega, i_\xi]fdz_1\wedge\cdots\wedge dz_k \\
&= \Omega i_\xi(fdz_1\wedge\cdots\wedge dz_k) + i_\xi\Omega(fdz_1\wedge\cdots\wedge dz_k) \\
&= \Omega(f)i_\xi(dz_1\wedge\cdots\wedge dz_k) + i_\xi(\Omega(f)\wedge dz_1\wedge\cdots\wedge dz_k) \\
&= \Omega(f)i_\xi(dz_1\wedge\cdots\wedge dz_k) \\
&+ (i_\xi\Omega(f))dz_1\wedge\cdots\wedge dz_k - \Omega(f)i_\xi(dz_1\wedge\cdots\wedge dz_k) \\
&= (i_\xi\Omega(f))dz_1\wedge\cdots\wedge dz_k \\
&= \left(i_\xi(\sum_s k_s(1-s)(f)\alpha_s^{-1}d\alpha_s)\right)dz_1\wedge\cdots\wedge dz_k \\
&= \sum_s k_s(1-s)(f)\alpha_s^{-1}\langle\alpha_s,\xi\rangle dz_1\wedge\cdots\wedge dz_k
\end{aligned}$$

Notice that in the third line, we used the fact that $i_\xi dz_i \in \mathbb{C}$, and that Ω is $\mathbb{C}$-linear

In summary:

$$\pi_\xi = \sum_s k_s\alpha_s^{-1}\langle\alpha_s,\xi\rangle(1-\hat{s}) \tag{A.1}$$

$\square$

Lemma A.0.17. *For $\xi \in V$, $x \in V^*$, we have*

$$[\tilde{\mathcal{L}}_\xi, m_x] = \left(\langle x,\xi\rangle - \sum_s k_s\langle x,\alpha_s^\vee\rangle\langle\alpha_s,\xi\rangle\hat{s}\right)$$

Proof. Now for $x, z_i \in V^* \subset A$ and $f \in A$ (writing just dz for $dz_1 \wedge \cdots \wedge dz_k$).

$$\begin{aligned}
[\pi_\xi, m_x](f dz) &= \pi_\xi(x f dz) - x\pi_\xi(f dz) \\
&= \sum_s k_s \left((1-s)(xf) - x(1-s)(f)\right) \alpha_s^{-1}\langle \alpha_s, \xi\rangle dz \\
&= \sum_s k_s \left(xf - {}^s x {}^s f - xf + x {}^s f\right) \alpha_s^{-1}\langle \alpha_s, \xi\rangle dz \\
&= \sum_s k_s (1-s)(x) {}^s f \alpha_s^{-1}\langle \alpha_s, \xi\rangle dz \\
&= \sum_s k_s \langle x, \alpha_s^\vee\rangle \alpha_s {}^s f \alpha_s^{-1}\langle \alpha_s, \xi\rangle dz \\
&= \left(\sum_s k_s \langle x, \alpha_s^\vee\rangle \langle \alpha_s, \xi\rangle \hat{s}\right)(f dz)
\end{aligned}$$

Which gives:

$$\begin{aligned}
[\tilde{\mathcal{L}}_\xi, m_x] &= [\mathcal{L}_\xi, m_x] - [\pi_\xi, m_x] \\
&= [\mathcal{L}_\xi, i_x] - [\pi_\xi, m_x] \\
&= \left(i_{[\xi,x]} - \sum_s k_s \langle x, \alpha_s^\vee\rangle \langle \alpha_s, \xi\rangle s\right) \\
&= \left(\langle x, \xi\rangle - \sum_s k_s \langle x, \alpha_s^\vee\rangle \langle \alpha_s, \xi\rangle \hat{s}\right)
\end{aligned}$$

□

Lemma A.0.18. *For $\xi \in V$, $x \in V^*$, we have*

$$[\tilde{\mathcal{L}}_\xi, m_{dx}] = 0$$

Proof. Here again for $x, z_i \in V^* \subset A$ and $f \in A$ (still writing just dz for $dz_1 \wedge \cdots \wedge dz_k$).

It is clear from (A.1) that $[\pi_\xi, m_{dx}] = 0$. Then

$$\begin{aligned}
[\tilde{\mathcal{L}}_\xi, m_{dx}] &= [\mathcal{L}_\xi, m_{dx}] - [\pi_\xi, m_{dx}] \\
&= [\mathcal{L}_\xi, m_{dx}] \\
&= \mathcal{L}_\xi(dx) \\
&= d\mathcal{L}_\xi(x) \\
&= d(\langle x, \xi \rangle) \\
&= 0
\end{aligned}$$

□

Lemma A.0.19. *For $\xi, \eta \in V$, we have*

$$[\tilde{\mathcal{L}}_\xi, i_\eta] = i_{[\xi,\eta]}$$

Proof. We have

$$\begin{aligned}
[\tilde{\mathcal{L}}_\xi, i_\eta] &= [\mathcal{L}_\xi - \pi_\xi, i_\eta] \\
&= [\mathcal{L}_\xi, i_\eta] - [\pi_\xi, i_\eta] \\
&= i_{[\xi,\eta]} - \pi_\xi i_\eta + i_\eta \pi_\xi \\
&= i_{[\xi,\eta]}
\end{aligned}$$

Because indeed, for 1-forms $f dz$, $z \in V^*$, for example we have:

$$\begin{aligned}
(\pi_\xi i_\eta - i_\eta \pi_\xi)(f dz) &= \pi_\xi(f\langle z, \eta \rangle) - i_\eta \left(\sum_s k_s {\alpha_s}^{-1} \langle \alpha_s, \xi \rangle (1-s)(f) dz \right) \\
&= (\langle z, \xi \rangle) \pi_\xi(f) - i_\eta \left(\sum_s k_s {\alpha_s}^{-1} \langle \alpha_s, \xi \rangle (1-s)(f) dz \right) \\
&= 0
\end{aligned}$$

□

Lemma A.0.20 (Commutators with $\hat{s}$)**.** *For $\xi \in V$, $x \in V^*$ and $s \in W$ we have*

$$[\hat{s}, m_{dx}] = 0$$

$$[\hat{s}, i_\xi] = 0$$

$$[\hat{s}, m_x] = \langle x, \alpha_s^\vee \rangle \hat{s} m_{\alpha_s}$$

$$[\hat{s}, \tilde{\mathcal{L}}_\xi] = \langle \alpha_s, \xi \rangle \hat{s} \tilde{\mathcal{L}}_{\alpha_s^\vee}$$

Proof. First we have (using similar notations for $dz = dz_1 \wedge \cdots \wedge dz_k, z_i \in V^*$)

$$\begin{aligned} [\hat{s}, i_\xi](f dz) &= \hat{s}(i_\xi f dz) - i_\xi(\hat{s}(f dz)) \\ &= ({}^s f)(i_\xi dz) - i_\xi({}^s f dz) \\ &= 0 \end{aligned}$$

where the second line comes from the fact that $i_\xi dz$ has constant coefficients of the form $\pm\langle z_i, \xi \rangle \in \mathbb{C}$.

Next, we have

$$\begin{aligned} [\hat{s}, m_x](f dz) &= \hat{s}(m_x f dz) - m_x(\hat{s}(f dz)) \\ &= {}^s f {}^s x dz - x {}^s f dz \\ &= -(x - {}^s x) {}^s f dz \\ &= -\langle x, \alpha_s^\vee \rangle \alpha_s \hat{s}(f dz) \end{aligned}$$

While it is clear that

$$[\hat{s}, m_{dx}] = 0$$

Next, using $sf = ({}^s f)s$ for $f \in A$ as multiplication operator on ΩA, and similarly for $\hat{s}$

$$\begin{aligned}
\hat{s}\pi_\xi &= \hat{s}(\sum_s k_s {\alpha_s}^{-1}\langle \alpha_s, \xi\rangle(1-\hat{s})) \\
&= \sum_s k_s({}^s{\alpha_s}^{-1})\langle \alpha_s, \xi\rangle \hat{s}(1-\hat{s}) \\
&= \sum_s k_s(-{\alpha_s}^{-1})\langle \alpha_s, \xi\rangle(\hat{s}-1) \\
&= \pi_\xi
\end{aligned}$$

While

$$\begin{aligned}
\pi_{{}^s\xi} &= \sum_s k_s {\alpha_s}^{-1}\langle \alpha_s, {}^s\xi\rangle(1-\hat{s}) \\
&= \sum_s k_s {\alpha_s}^{-1}\left\langle \alpha_s, \xi - \langle \alpha_s, \xi\rangle \alpha_s^\vee\right\rangle(1-\hat{s}) \\
&= \sum_s k_s {\alpha_s}^{-1}(\langle \alpha_s, \xi\rangle - \langle \alpha_s, \xi\rangle\langle \alpha_s, \alpha_s^\vee\rangle)(1-\hat{s}) \\
&= \sum_s k_s {\alpha_s}^{-1}(-\langle \alpha_s, \xi\rangle)(1-\hat{s}) \\
&= \sum_s k_s {\alpha_s}^{-1}\langle \alpha_s, \xi\rangle(\hat{s}-1) \\
&= \sum_s k_s {\alpha_s}^{-1}\langle \alpha_s, \xi\rangle(1-\hat{s})\hat{s} \\
&= (\pi_\xi)\hat{s}
\end{aligned}$$

Which gives

$$\hat{s}\pi_\xi\hat{s} = \pi_{{}^s\xi}$$

Finally it is an easy computation that for $\xi \in V$ a *constant* vector field, $\hat{s}\mathcal{L}_\xi\hat{s} = \mathcal{L}_{{}^s\xi}$ so that

$$\hat{s}\tilde{\mathcal{L}}_\xi\hat{s} = \tilde{\mathcal{L}}_{{}^s\xi}$$

And thus

$$\begin{aligned}[\hat{s}, \tilde{\mathcal{L}}_\xi] &= \hat{s}\tilde{\mathcal{L}}_\xi - \tilde{\mathcal{L}}_\xi \hat{s} \\ &= \hat{s}\tilde{\mathcal{L}}_\xi - \hat{s}\tilde{\mathcal{L}}_{^s y} \\ &= \hat{s}\tilde{\mathcal{L}}_{\xi - {^s\xi}} \\ &= \hat{s}\langle \alpha_s, \xi \rangle \tilde{\mathcal{L}}_{\alpha_s^\vee}\end{aligned}$$

□

Lemma A.0.21 (Commutators of Dunkl operators)**.** *For $\xi, \eta \in V$, the relation $[\tilde{\mathcal{L}}_\xi, \tilde{\mathcal{L}}_\eta] = 0$ still holds when $\tilde{\mathcal{L}}_\xi$ and $\tilde{\mathcal{L}}_\eta$ are viewed as differential operators on forms.*

Proof. This follows immediately from the fact that for $f \in A$, $z_1, \dots, z_k \in V^*$

$$\tilde{\mathcal{L}}_\xi(f dz_1 \wedge \cdots \wedge dz_k) = T_\xi(f) dz_1 \wedge \cdots \wedge dz_k$$

where T_ξ is the classical Dunkl operator operating on $\mathbb{C}[V]$. □

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